The WDVV equations in Seiberg-Witten theory

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THE WDVV EQUATIONS IN SEIBERG-WITTEN THEORY

PROEFSCHRIFT

ter verkrijging van
de graad van doctor aan de Universiteit Twente,
op gezag van de rector magnificus,
prof.dr. F.A. van Vught,
volgens besluit van het College voor Promoties
in het openbaar te verdedigen
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door

Lukas Korneel Hoevenaars
geboren op 17 juni 1975
te Nijmegen

Dit proefschrift is goedgekeurd door de promotor

prof.dr.ir. R. Martini

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Voorwoord

In de Bommelverhalen van Marten Toonder komen twee wetenschappers voor: de gewetenloze en machtswellustige professor Joachim Sickbock en de van oorsprong Duitse stadsgeleerde professor Zbygniew Prlwytzkofski. Om hun karakters te illustreren zal ik enkele stukjes citeren uit het verhaal "De zonnige kijk", waarin professor Sickbock een mislukte poging doet om een kunstmatig intelligente levensvorm te ontwikkelen:

> 'Alles is gereed,' sprak hij tot zichzelf. 'Daar ligt het chemisch element uit de vierde groep van het periodiek systeem te wachten op de hyper-ontlading, die het zal omvormen tot de monade, die mij voor de geest zweeft...'
>
> ...
>
> 'Het is ergerlijk, dat men mij maar laat tobben zonder subsidie, terwijl ik de aarde zou kunnen bevolken met een voorgevormde levensvorm.'
>
> ...
>
> 'Glas,' mompelde hij vol afschuw. 'Mijn proef is mislukt! In plaats van een brein heb ik glas gemaakt. En mijn transmutator is vernietigd, zodat ik het niet kan overdoen. Ei, hoe armzalig is het leven van een miskend geleerde...'

De arme professor Sickbock moet voor de financiering van zijn onderzoek keer op keer een beroep doen op de derde geldstroom (Olivier Bommel) en gaat daarbij zelfs zover dat hij zijn onderzoek ombuigt naar een voor hem totaal oninteressante richting, in dit geval het maken van goud. Dit dient alleen om in de uren die overschieten zich te kunnen richten op zijn werkelijke onderzoek. Dit is tegenwoordig ook in het werkelijke leven het lot van vele wetenschappers.

Overigens doet dit citaat mij denken aan de verarming van de huidige wetenschappelijke taal, met name in theoretische hoge-energie fysica. Binnen deze tak van wetenschap wordt het weinig toevoegende prefix *hyper*, dat in het citaat hierboven voorkomt, veelvuldig gebezigd. Maar wat erger is, men kan in dezelfde zin het prefix *super* tegenkomen wat notabene hetzelfde betekent. Zo ontstaan uitdrukkingen als 'het hypermultiplet van een supersymmetrische quantumveldentheorie'. Het gebruik van deze twee prefixen had mijns inziens vermeden moeten worden, maar ik vrees dat het kwaad al is geschied.

In contrast met Sickbock is daar de Rommeldamse stadsgeleerde Prlwytzkofski, die in hetzelfde verhaal ook voor het voetlicht treedt:

> Professor Prlwytzkofski zat in het laboratorium de krant te lezen om zich te ontspannen na de doorwerkte nacht. Maar veel vermaak putte hij er niet uit. 'Praw,' mompelde hij. 'Door nieuwer gouddekking meer geld vervoegbaar. Afkoeling van financieel klimaat. Welk ener kinderij. Alsof geld wichtiger is, dan duchtiger wetenschappelijker arbeid!'

Merk vooral de keuze op van het lidwoord in *de doorwerkte nacht*, suggererend dat dit meer dan eens per week voorkomt. Het is duidelijk dat Prlwytzkofski een vaste betrekking heeft en zich dus de luxe kan permitteren om zich toe te leggen op zuiver wetenschappelijk onderzoek. Zonder enige andere motivatie dan zijn eigen nieuwsgierigheid kan hij onpartijdig onderzoek doen en dat is natuurlijk hoe het zou moeten zijn voor iedere wetenschapper.

Ikzelf voel mij meer verwant met Prlwytzkofski dan met Sickbock, en in dat licht vind ik het prettig dat ik mezelf gedurende de afgelopen vijf jaar AIO (en geen OIO) heb mogen noemen. We leven nu echter in een tijdsgewricht waarin de Sickbocks oprukken, het geld het wint van de nieuwsgierigheid en dientengevolge het zuivere onderzoek met uitsterven wordt bedreigd. Ei, ei.

De afgelopen vijf jaar die ik aan mijn promotie heb besteed zijn in meerdere opzichten avontuurlijk geweest. Het is allemaal begonnen toen mijn vriend en wiskundestudent Rob me voorstelde aan prof. Ruud Martini, die juist op dat moment op zoek was naar een fysicus met de juiste kennis. Ik kan me nog herinneren dat de ongedwongen sfeer binnen de groep voor mij een van de grootste aantrekkingspunten vormde om in Enschede te beginnen. Professor Martini bleek gelukkig ook van het Prlwytzkofski-soort en met name door zijn liefde voor het vak en zijn goede begeleiding is hij voor mij een groot voorbeeld geweest.

Tevens gaat mijn dank uit naar Paul Kersten, die de begeleiding van Ruud overnam toen hij ziek was. Hij heeft zich meer dan gewetensvol van die taak gekweten en ik weet zeker dat dit proefschrift er niet zou hebben gelegen zonder zijn hulp en vriendschap. Dankzij hem heb ik ook contacten kunnen leggen met het ITEP in Moskou en ik bedank met name Andrei Marshakov, Alexei Morozov, Andrei Mironov en Joseph Krasil'shchik voor de plezierige en nuttige maand die ik in Moskou heb besteed.

Al mijn collega's hebben gezorgd voor een prettige werksfeer, maar een aantal van hen wil ik toch met name noemen. Ik bedank Gerard en Gerhard voor de prettige sfeer en de gezellige spelletjesavonden en Jeroen, Jan-Kees, Johann, Eugene, Gerhard en Gerard voor hun pogingen mij te leren bridgen in de lunchuurtjes. Diana stond altijd klaar voor secretariële ondersteuning en een gezellige babbel. Sergei en Johan hebben lijdzaam mijn aanwezigheid als kamergenoot ondergaan, hulde daarvoor.

Ik kan me nog herinneren dat Rob en ik al in de eerste maanden van mijn verblijf in Twente uitrekenden wie er eerder klaar zou zijn: ik met mijn promotie of hij met zijn studie. Aangezien hij al twee jaar bezig was, waren we het snel met elkaar eens. Gedurende de jaren hebben we die prognose echter wat moeten aanpassen links en rechts... En nu dan ligt hier mijn proefschrift, na vijf jaar van hervonden vriendschap. Maar ik verheug me nu al op jouw afstuderen en op de vele jaren vriendschap die we hopelijk zullen blijven delen.

Ik bedank ook al mijn vrienden en mijn ouders en zussen, die zich al die tijd hebben afgevraagd waar ik in 's hemelsnaam mee bezig was. Nou, jullie kunnen het nu lezen en je vergapen aan al die ingewikkelde wiskunde. Maar belangrijker voor me is dat jullie mij begrijpen als mens en me altijd hebben gesteund in wat ik doe.

Tot slot is daar Linda, mijn wonder op wielen, voor wie woorden slechts tekort kunnen schieten. Dankjewel voor alles.

Introduction

Partial differential equations (PDE's) come in various sorts, classified for example according to the order, the number of variables, whether the equations are linear or nonlinear, and whether they are overdetermined, underdetermined or well-determined. A general theory to find explicit solutions of PDE's does not exist for any substantial class of equations, but results can be obtained for restricted classes. A general result about existence of solutions of PDE's is the Cauchy-Kovalevskaya theorem, which in principal opens the possibility to construct solutions of an initial value problem for the class of well-determined PDE's, in terms of power series. This theorem can be applied regardless of the order, the number of variables or the (non)linearity. This theorem covers many important PDE's arising from physics, for example evolution equations (the heat equation, the KdV equation), the Laplace equation, Maxwell's equations and the Navier-Stokes equations.

In this thesis we consider a system of equations called the Witten-Dijkgraaf-Verlinde-Verlinde or WDVV system. In the year 1991 it appeared in physics, more particularly in the studies of two-dimensional conformal field theory, where it was discovered by Witten [62] and Dijkgraaf, E. Verlinde and H. Verlinde [16]. Roughly speaking, this system expresses the condition for the third order derivatives of a function to be the structure constants of an associative, commutative algebra with a unit. The WDVV system defies a treatment with the Cauchy-Kovalevskaya theorem: it is a system of highly overdetermined[1] third order nonlinear equations which can be defined for an arbitrary number of variables greater than or equal to three. In fact, because of the overdeterminedness one might expect that no solutions exist at all. One indication that the WDVV system is very special, is that it does indeed admit exact solutions for any number of variables. For example, in one of the articles in which the WDVV system made its first appearance [16], the authors gave a class of solutions for any number of variables. These solutions are polynomial and were given for the finite dimensional root systems of ADE type. Denoting the root space by V and the Coxeter group by W, the variables then correspond to coordinates on the space of Coxeter orbits V/W.

Within the mathematical community, the first to study the WDVV equations intensively was Boris Dubrovin. Among other things, he has shown that the polynomial solutions can be defined for any finite Coxeter group and he related them with the unfoldings of isolated singularities [19]. Moreover, he has shown that the WDVV equations admit a representation in terms of a socalled zero-curvature form with a spectral parameter, see for example the survey article [18]. This is an indication that the system is within a special class of PDE's: the class of integrable systems. There is a lot of confusion in the literature about the concept of integrability, involving zero-curvature representations, conservation laws, symmetries, bihamiltonian structures, the Painlevé test and so on. This topic will not be discussed in any detail here. Among PDE's, integrable systems are an exception in the sense that methods exist to discuss explicit solutions. This then gives another reason to consider the WDVV equations as something special.

A third reason to study the WDVV equations is that in the other article in which the equations made their first appearance [62] it was conjectured that a certain function, appearing in the

1 The only exception is the WDVV equation for three variables, which is well-determined. In this case Cauchy-Kovalevskaya can indeed be applied [18].

theory of intersection numbers on the moduli spaces of curves, is a tau function of the KdV hierarchy. This was later proven to be true by Kontsevich [36]. This function has a power series expansion, with the zero order term satisfying the WDVV equations. It was later shown that this zero order term itself is a tau function of the dispersionless KdV hierarchy. This establishes a link between the WDVV equations and a well-known integrable hierarchy.

In the years after 1991, the role of the WDVV equations in mathematics became important in enumerative geometry, within the context of quantum cohomology and Gromov-Witten invariants, which are topological invariants of symplectic manifolds. In particular, the article [62] considers the quantum cohomology of a single point. As another example, we mention that it was shown [37] that the problem of finding the number of rational curves of degree k passsing through $3k-1$ generic points in the complex projective plane $\mathbf{P}^2$ is solved in terms of a generating function F depending on 3 variables. This generating function satisfies the WDVV equations.

In this thesis we will not go into the solutions of the WDVV equations coming from singularity theory, nor solutions coming from quantum cohomology and Gromov-Witten invariants. The main motivation for this choice is that in 1996, a generalized version of the WDVV equations was introduced by Marshakov, Mironov and Morozov [45] and it is this generalized WDVV system that constitutes the main topic of this thesis. The physical context in which these equations were found is called $\mathcal{N}=2$ supersymmetric Yang-Mills theory, also called Seiberg-Witten theory [59]. The generalized system is truely a generalization of the original system, in the sense that solutions to the original equations are automatically solutions to their generalized counterparts but the converse statement is false. The generalized WDVV equations still retain many of the properties that make the study of the original equations worthwhile. For example we consider in this thesis constructions of explicit functions for any number of variables, which are solutions of the generalized equations but not of the original ones. Moreover, these solutions were shown to be tau functions for the Whitham hierarchy corresponding to the periodic Toda chain, thus establishing a link between generalized WDVV equations and known integrable hierarchies. Finally, the generalized equations themselves are indicated to be integrable since for example they have a zero-curvature representation (albeit without a spectral parameter) and they allow the construction of explicit solutions.

The introduction of the original and generalized WDVV equations together with some of their background in physics as well as mathematics is the topic of the first chapter. The second paragraph of this chapter contains a discussion of a continuous group of contact symmetries which will be used in chapter three. These are symmetries of the generalized, but not of the original equations and they have a clear physical meaning as electro-magnetic duality transformations.

We then come to the main topic of this thesis, which is to discuss explicit constructions of solutions of the generalized WDVV system coming from four and five-dimensional physics. Typically these solutions, called prepotentials, can be expressed as an infinite power series in an auxiliary parameter and the zero order term separately satisfies the equations. Such zero order terms are called perturbative prepotentials. The construction of the perturbative prepotentials is as follows: one starts with a base function $f(x)$, which depends on only one variable. Then for every rank N root system R with root space V one considers the function

$$\mathcal{F}(a_1, ..., a_N) = \sum_{\alpha \in R} f((\alpha, a))$$

Here the sum is over all roots α, the element $a = \sum_i \alpha_i a_i \in V$ is expressed on a basis of simple roots and $(.,.)$ is the standard Euclidean inner product on V. Taking the four-dimensional base function $f_4(x) = \frac{1}{2}x^2 \log(x)$ we prove that $\mathcal{F}$ satisfies the generalized (but not the original) WDVV system for any root system R. The five-dimensional case is more problematic. Taking its base function

$$f_5(x) = \frac{1}{6}x^3 - \frac{1}{4}Li_3(e^{-2x}) = \frac{1}{6}x^3 - \frac{1}{4}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^3}$$

we show that the corresponding perturbative prepotential does not satisfy the WDVV system. Due to a result in string theory, we are led to consider adding a cubic polynomial to the five-dimensional perturbative prepotential. Keeping some free parameters in the cubic polynomial leads only to partial succes, since solutions can now be found for certain combinations of the parameters, but the relation between the solutions and the Coxeter groups we started from is lost. This is because a generic Coxeter group does not contain a cubic invariant polynomial. A careful study of the proof of the four-dimensional case leads us to add a cubic polynomial involving a new variable a_0 in such a way that the new prepotential is

$$\mathcal{F}(a_0, ..., a_N) = \sum_{\alpha \in R} f_5((\alpha, a)) + \gamma\left[\frac{1}{6}a_0^3 + \frac{1}{2}a_0(a, a)\right]$$

This prepotential is Coxeter invariant and contains only one parameter γ. We show that for each crystallographical root system it is possible to find one value of γ such that the corresponding prepotential satisfies the generalized WDVV system. Moreover, somewhat surprisingly it even satisfies the original WDVV equations. To the best of our knowledge these solutions to the original WDVV equations are new, and their construction is quite nontrivial since the addition of a polynomial involving a new variable is usually expected to spoil the WDVV equations. This discussion of the four and five-dimensional perturbative prepotentials is the subject of chapter two.

Chapter three deals with the construction of the full (as opposed to the perturbative) prepotentials, which is much more complicated. In this thesis we restrict ourselves to the four-dimensional case. The starting point is the Seiberg-Witten data, consisting of three ingredients. The first ingredient is a particular family of Riemann surfaces associated with any simple Lie algebra $\mathfrak{g}$. The moduli u_i of the family are given by Weyl invariant polynomials on the Cartan subalgebra of $\mathfrak{g}$. The second ingredient is a special meromorphic differential λ_{SW}, such that its first order derivatives with respect to the moduli are holomorphic forms on the Riemann surface. The third ingredient is a special set of $2N$ out of the possible $2g$ cycles on the Riemann surface, where g denotes its genus. These special cycles are denotes by $A_1, ..., A_N, B_1, ..., B_N$ and satisfy the usual intersection relations.

The next step is to make a change of variables from the moduli u_i to the new variables

$$a_i = \oint_{A_i} \lambda_{SW}$$

which are period integrals of λ_{SW} over the subset of special cycles of type A. The period integrals over the other special cycles $b_i = \oint_{B_i} \lambda_{SW}$ can be differentiated with respect to the a_i giving

$$\Pi_{ij} = \frac{\partial b_i}{\partial a_j} = \oint_{B_i} \frac{\partial \lambda_{SW}}{\partial a_j}$$

The matrix Π_{ij} can be shown to be a submatrix of the period matrix of the Riemann surface, hence it is symmetric. As a result the objects b_i can be integrated locally to a single function $\mathcal{F}(a_1, ..., a_N)$ which is called the prepotential.

We then go on to construct an associative, commutative[2] algebra with unit out of the first order derivatives $\omega_i = \frac{\partial \lambda_{SW}}{\partial a_i}$ with structure constants C_{ij}^k. Recall that in order to show that the prepotential $\mathcal{F}$ satisfies the WDVV equations, it is necessary prove that its third order derivatives give an associative, commutative algebra with unit. Identifying this algebra with the algebra of forms leads to a relation between the third order derivatives of $\mathcal{F}$ and the structure constants C_{ij}^k. In this thesis, we consider two methods of proving this relation but here we will sketch only one.

The WDVV equations are equivalent to the following relation between structure constants and third order derivatives $\mathcal{F}_{ijk}$

$$\mathcal{F}_{ijk} = \sum_{l,m=1}^{N} C_{ij}^m \alpha_l \mathcal{F}_{klm}$$

where the α_l are a set of (possibly a-dependent) parameters. Since the first order derivatives of $\mathcal{F}$ can be given in terms of period integrals of λ_{SW}, these equations become a set of second order linear equations which have to be satisfied by λ_{SW}.

Basically, proving the relation between structure constants and third order derivatives is now reduced to finding solutions to a second order system of PDE's known as a Picard-Fuchs system. Such a reduction, from a third order to a second order system, can also be done in a trivial way: since the WDVV equations for a function F are homogeneous of order three, one can also rewrite them as a system of second order equations on the first order derivatives of F. In that case solving the equations means solving the second order system in terms of N solutions F_i together with the condition that the F_i integrate to a single function F. The main advantages of the construction of solutions described above is that we consider a second order linear system (the Picard-Fuchs system), which can be rewritten in terms of a higher order ODE with regular singular points. The standard theory of ODE's can then be used to show that there exist precisely N independent solutions $F_i = \oint_{B_i} \lambda SW$. Moreover, the possibility of integrating these solutions to a single function is guaranteed. This then proves that the prepotentials satisfy the WDVV equations, which is the main result of chapter three.

Another topic in chapter three is the relation between the Seiberg-Witten prepotentials and an integrable system called the periodic Toda chain. This is a dynamical system of N particles on a chain with interactions that can be defined for any rank N simple Lie algebra $\mathfrak{g}$. For such systems, the classical notion of integrability involves the existence of precisely N conserved quantities in involution, so that one can make a change of coordinates to action-angle variables in terms of which the flow of the system linearizes on the Liouville torus. For the Toda system, the conserved quantities can be obtained from a Lax representation. Given a Lax representation with spectral parameter z, the spectrum of the Lax operator $A(z)$ is invariant under the flow of the system. This spectrum is given by the characteristic polynomial

$$det\,[A(z) - x \cdot I] = 0$$

This gives a family of Riemann surfaces, depending on the coordinates and momenta of the particles, which is also invariant under the flow. It so happens that this family of Riemann

2 One takes the tensor product of holomorphic forms instead of the more common wedge product, which makes the algebra commutative.

surfaces is the same as the one occurring in Seiberg-Witten theory. Moreover, the Seiberg-Witten differential λ_{SW} turns out to be the action differential pdq of the Toda chain. It can be shown that the flow of the system linearizes on the Jacobian of the Riemann surfaces. Consideration of dimensions suggests that the Jacobian contains a subvariety which can be identified with the Liouville torus. Indeed the Seiberg-Witten differential together with the special cycles select a $2N$-dimensional subvariety which plays this role and is called the distinguished Prym variety. Moreover, the prepotential itself can be identified with a tau function of the socalled Whitham hierarchy associated with the periodic chain. These relations between the Seiberg-Witten prepotentials and the periodic Toda chain will be discussed in some detail in chapter three.

Finally, in the case of Lie algebra A_N we show how the prepotential can be expressed as an infinite power series in an auxiliary parameter and the zero order term is shown to be identical to the perturbative prepotential considered in chapter two. By this time, we have two independent proofs that the perturbative prepotential satisfies the WDVV system: we have a direct check in chapter two and we have shown that it is the zero order term of a function that has been shown to satisfy the WDVV system in chapter three.

Chapter 1

The WDVV equations

Chapter 1

In this chapter we start by introducing the original and generalized WDVV equations together with some of their physical and mathematical background. We then discuss the existence of a continuous group of symmetries of the generalized system which will be used in chapter three. Finally, we discuss the possibility to create a coordinate invariant formulation of the generalized WDVV equations along the lines of Dubrovin's work for the original system, which leads to the concept of Frobenius manifolds. In particular, we explain why the simplest attempt towards such a coordinate invariant formulation must fail for the generalized system.

1.1 The original WDVV equations

The original Witten-Dijkgraaf-Verlinde-Verlinde equations were put forward in [62], [16]. They form a system of third order nonlinear partial differential equations for a function F of N variables.

Definition 1.1. *Consider a function $F(t_0, ..., t_{N-1})$ and matrices*

$$[F_i]_{jk} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} \tag{1.1.1}$$

The original WDVV equations are given by

$$F_i F_0^{-1} F_j = F_j F_0^{-1} F_i \qquad\qquad i,j = 0, ..., N-1 \tag{1.1.2}$$

where the matrix F_0 is required to be constant (t_i independent) and invertible. This makes the variable t_0 a special one.

The system (1.1.2) is trivially satisfied for $N = 1, 2$ but extremely difficult to solve for higher N. Nevertheless, we will find that this highly overdetermined[1] system admits exact solutions for all N. But before coming to the discussion of solutions, let us reinterpret the WDVV system in terms of families of associative algebras. We introduce the objects

$$C_{ij}^k = \sum_{m=0}^{N-1} F_{ijm} \left[F_0^{-1}\right]^{mk} \tag{1.1.3}$$

which are symmetric in i, j and in general t-dependent. In terms of the matrices $[C_i]_j^k = C_{ij}^k$ the WDVV system expresses the commutation of C_i and C_j

$$[C_i, C_j] = 0 \qquad\qquad i,j = 0, ..., N-1 \tag{1.1.4}$$

1 The number of independent third order derivatives grows with N^3 while the number of independent algebraic relations between them grows as N^4.

Regarding the C_{ij}^k as structure constants of an algebra

$$\phi_i\phi_j = \sum_k C_{ij}^k \phi_k \tag{1.1.5}$$

we find that this algebra is commutative due to the symmetry in i, j and associative because of (1.1.4). Moreover, the algebra has a unit ϕ_0 because $C_{0j}^k = \delta_{jk}$. Since this is true for all values of the t_i the structure constants in fact form a family of associative, commutative algebras with unit. We therefore have the following

Proposition 1.2. *A function $F(t_0, ..., t_{N-1})$ satisfies the original WDVV equations if and only if*

- *the matrix F_0 (consisting of third order derivatives) is constant*
- *there exists an N-parameter family of commutative, associative algebras with unit, whose structure constants C_{ij}^k are related to F through (1.1.3)*

This alternative definition of the WDVV equations is why they are also called associativity equations in the literature. The WDVV system is studied in many different contexts, and often the family of algebras gets a natural interpretation there.

1.1.1 Example of a solution

The physical context in which the WDVV equations made their first appearance is a two-dimensional topological $\mathcal{N} = 2$ superconformal field theory [16]. Its Hilbert space of states is finite dimensional and one of the main objects in the theory is the so-called superpotential $W(x)$ which in a particular example is a monomial

$$W(x) = x^{N+1} \tag{1.1.6}$$

Quantum effects perturb the superpotential to[2]

$$W(x) = x^{N+1} + u_{N-1}x^{N-1} + ... + u_0 \tag{1.1.7}$$

and the physical Hilbert space is $\mathbf{C}[x]/I$ where I is the ideal in $\mathbf{C}[x]$ generated by $W' = \frac{\partial W}{\partial x}$. Through an operator-state correspondence, the elements of the Hilbert space ϕ_i are also considered to be operators, and one expects an operator algebra

$$\phi_i\phi_j = \sum_k C_{ij}^k \phi_k \tag{1.1.8}$$

since the repeated operator $\phi_i\phi_j$ is again an element of the finite-dimensional Hilbert space. Indeed, by identifying the ϕ_i with representatives $\frac{\partial W}{\partial u_i} = x^i$ of the basis elements of the quotient $\mathbf{C}[x]/I$ we find that they span the quotient algebra. Note that this is an N-parameter family of commutative, associative algebras with unit $\phi_0 = x^0 = 1$.

2 In mathematical terms, (1.1.7) is an unfolding of the type A_N singularity

We now introduce a nondegenerate bilinear form on $\mathbf{C}[x]/I$ through

$$(\phi, \chi) = \frac{1}{2\pi i} \oint_{W'=0} \frac{\phi\chi}{W'} dx \tag{1.1.9}$$

This bilinear form is independent of the representatives chosen for the equivalence classes, since terms containing W' clearly don't contribute to the residue. We can use it to raise and lower indices of C_{ij}^k and define

$$C_{ijm} = \frac{1}{2\pi i} \oint_{W'=0} \frac{\phi_i \phi_j \phi_m}{W'} = \sum_k C_{ij}^k (\phi_k, \phi_m) \tag{1.1.10}$$

It follows that $C_{0ij} = (\phi_i, \phi_j)$. Since C_{ijm} is totally symmetric in its indices, one may wonder if there exists a function $F(u_0, ..., u_{N-1})$ such that

$$\begin{aligned} C_{ijm} &= \frac{\partial^3 F}{\partial u_i \partial u_j \partial u_m} \\ C_{0ij} &= \text{constant} \end{aligned} \tag{1.1.11}$$

If such a function indeed exists, then it satisfies the WDVV system due to proposition 1.2. However, it is easily checked that neither equation in (1.1.11) holds. The way to define a function which does satisfy the WDVV equations is to consider a change of variables from u_i to a particular set t_i which depend polynomially on the u_i. We mention the next result without proof.

Proposition 1.3. *[16] There exists a set of variables t_i depending polynomially on the u_i with the following properties: the derivatives $\phi_i = \frac{\partial W}{\partial t_i}$ are a good set of generators for the ring $\mathbf{C}[x]/I$, and the corresponding structure constants $C_{ijm}(t)$ are the third order derivatives of a function $F(t_0, ..., t_{N-1})$ with respect to the t variables. Moreover, the nondegenerate bilinear form whose matrix elements are C_{0ij} is constant. Therefore the function $F(t)$ satisfies the original WDVV system (1.1.2).*

In the physical context, this function F plays the role of the free energy of the model.

Example 1.4. *Among the models with superpotential (1.1.7) we give the free energy for the one with $N = 3$*

$$F(t_0, t_1, t_2) = \frac{1}{2} t_0^2 t_1 + \frac{1}{2} t_0 t_2^2 + \frac{1}{4} t_1^2 t_2^2 + \frac{1}{60} t_1^5 \tag{1.1.12}$$

The free energies for other N are also polynomial.

1.1.2 Mathematical background

From a mathematical viewpoint, there is a number of reasons why the WDVV equations are important. First of all a conjecture of Witten [62], later proven by Kontsevich [36], states that the free energy for certain 2-dimensional superconformal field theories coincides with the logarithm of a particular tau function of the KdV hierarchy. Moreover, one can generalize the mathematical setting[3] in such a way that the structure constants C_{ij}^k form the

3 In the above setting X is simply a point.

structure constants of a deformed or 'quantum' version of the cohomology ring of a compact symplectic manifold X. The function F in those cases contains information about nontrivial topological invariants of X, called Gromov-Witten invariants. In fact, F is a generating function for these invariants.

Example 1.5. *As an example, we consider the quantum cohomology of the complex projective line $\mathbf{P}^1$ and plane $\mathbf{P}^2$ [37]. The quantum cohomology ring for any $\mathbf{P}^d$ is given by $\mathbf{C}[x]/I$ where the ideal I is generated by $x^{d+1} - e^{-t_1}$. In the case of $\mathbf{P}^1$, the function F satisfies the WDVV equations trivially since it depends only on two variables*

$$F(t_0, t_1) = \frac{1}{2} t_0^2 t_1 + e^{t_1} \tag{1.1.13}$$

The first nontrivial case is $\mathbf{P}^2$, whose function F takes the form of a power series

$$F(t_0, t_1, t_2) = \frac{1}{2} t_0 t_1^2 + \frac{1}{2} t_0^2 t_2 + \sum_{n=1}^{\infty} \frac{N_n t_2^{3n-1}}{(3n-1)!} e^{n t_1} \tag{1.1.14}$$

This function F satisfies the WDVV equations if and only if the following relation holds

$$F_{222} = F_{112}^2 - F_{111} F_{122} \tag{1.1.15}$$

From this relation, one can find the numbers N_n recursively

$$N_n = (3n-4)! \sum_{a+b=n} \frac{a^2 b(3b-1)(2a-b)}{(3a-1)!(3b-1)!} N_a N_b \tag{1.1.16}$$

and the first few of them are

$$\begin{aligned} N_1 &= 1 \\ N_2 &= 1 \\ N_3 &= 12 \\ N_4 &= 620 \\ N_5 &= 87304 \end{aligned} \tag{1.1.17}$$

N_k receives the interpretation of the number of rational curves of degree k on $\mathbf{P}^2$ going through $3k-1$ generic points, thus F encodes topological data of $\mathbf{P}^2$. If we rewrite F in the form

$$F(t_0, t_1, t_2) = \frac{1}{2} t_0 t_1^2 + \frac{1}{2} t_0^2 t_2 + t_2^{-1} \phi(t_1 + 3\log(t_2)) \tag{1.1.18}$$

then it is proven in [18] that ϕ and therefore F actually converges for
Re$(t_1 + 3\log(t_2)) < \log\left(\frac{6}{5}\right)$.

1.1.3 Integrable structure and the deformed Euclidean connection

Yet another reason to study the WDVV equations is that they are equivalent to the compatibility conditions of a linear first order system with a parameter [18]. Such a realization is considered to be a strong indication of the integrability of the system, thus making it worthwhile to be studied. Consider the first order linear system

$$\left(\frac{\partial}{\partial t_i} + zC_i\right)\psi = 0 \qquad i = 0, ..., N-1 \tag{1.1.19}$$

where z is an arbitrary parameter, ψ is an N-dimensional vector of functions and the matrices $[C_i]_j^k$ satisfy the following restrictions:

- The matrix C_0 equals the $N \times N$ identity matrix
- All matrices C_i are t_0 independent
- There exists a constant matrix K such that

$$F_{ijm} := \sum_k C_{ij}^k K_{km} \tag{1.1.20}$$

  are totally symmetric in i, j, m.

We then have the following result [18]

Proposition 1.6. *The compatibility conditions of the system (1.1.19) are equivalent to the WDVV equations (1.1.2).*

Proof. The compatibility conditions are that $\partial_i\partial_j\psi = \partial_j\partial_i\psi$. Writing this out we get the following equation

$$(\partial_i C_j - \partial_j C_i)\, z + [C_i, C_j]\, z^2 = 0 \tag{1.1.21}$$

This second degree polynomial in z has to vanish identically, and since K is constant the first order term ensures existence of a function F whose third order derivatives are F_{ijm}. It then follows from (1.1.20) that the matrix $F_0 = K$ of third order derivatives is constant. The second order term in (1.1.21) then states that the function F satisfies the original WDVV equations. □

As usual, we can reformulate the compatibility conditions as the zero-curvature conditions of a connection. We introduce the deformed Euclidean connection $\tilde{\nabla}$ on $\mathbf{C}^N$ in terms of the coordinates t_i as follows

$$\tilde{\nabla}_i \frac{\partial}{\partial t_j} = \frac{\partial^2}{\partial t_i \partial t_j} + z\sum_k C_{ij}^k \frac{\partial}{\partial t_k} \tag{1.1.22}$$

The compatibility conditions or zero-curvature relations then read

$$\left[\tilde{\nabla}_i, \tilde{\nabla}_j\right] = 0 \tag{1.1.23}$$

The flat coordinates, in terms of which the covariant derivative with respect to $\tilde{\nabla}$ is just the ordinary derivative, are given by a set of N independent solutions of

$$\left(\frac{\partial^2}{\partial t_i \partial t_j} - \sum_k zC_{ij}^k \frac{\partial}{\partial t_k}\right)\chi = 0 \tag{1.1.24}$$

Note how the solutions of the first order linear system (1.1.19) as well as the second order system (1.1.24) have a known dependence on t_0. In fact, since $C_0 = I$ we find the simple behaviour

$$\begin{aligned} \psi(t_0, ..., t_{N-1}) &= e^{-zt_0}\hat{\psi}(t_1, ..., t_{N-1}) \\ \chi(t_0, ..., t_{N-1}) &= e^{zt_0}\hat{\chi}(t_1, ..., t_{N-1}) \end{aligned} \tag{1.1.25}$$

Making a Fourier transform in the t_0 variable we find that the WDVV equations are still equivalent to the compatibility conditions of the first order linear system

$$(\partial_i + C_i\partial_0)\,\psi = 0 \tag{1.1.26}$$

After this Fourier transform, the deformed Euclidean connection is no longer a connection. Nevertheless, the second order equations

$$\left(\frac{\partial^2}{\partial t_i \partial t_j} - \sum_k C_{ij}^k \frac{\partial^2}{\partial t_k \partial t_0}\right)\chi = 0 \tag{1.1.27}$$

will play an important role in the rest of this thesis, see for example section 3.3.3.

1.2 The generalized WDVV equations

In this section we introduce the main topic of this thesis, the generalized WDVV system of third order nonlinear partial differential equations. Furthermore we provide some physical and mathematical background to indicate why it is interesting to study these equations. Finally, we discuss a continuous group of symmetries.

The generalized WDVV equations were introduced in [45]

Definition 1.7. *Consider[4] a function $F(a_1, ..., a_N)$ and matrices*

$$\begin{aligned} [F_i]_{jk} &= \frac{\partial^3 F}{\partial a_i \partial a_j \partial a_k} \\ K_{ij} &= \sum_q \alpha_q F_{qij} \end{aligned} \tag{1.2.1}$$

Here the α_q are possibly a-dependent, and they are chosen in such a way that K is invertible for generic values of the variables a_i. The generalized WDVV equations is given by

$$F_i K^{-1} F_j = F_j K^{-1} F_i \qquad i, j = 1, ..., N \tag{1.2.2}$$

Remark 1.8. *The original WDVV equations require the existence of a special variable a_0 and a set of constants $\alpha_q = \delta_{q,0}$ such that $K = \sum_q \alpha_q F_q = F_0$ is a constant invertible matrix. For the generalized WDVV system we give up constancy of both α_q and K and therefore K^{-1} need not exist for all a_i .*

4 To distinguish between the original and generalized WDVV equations we switch from variables t_i to variables a_i.

We can reformulate the generalized WDVV equations in terms of a family of associative, commutative algebras. We introduce the objects

$$C_{ij}^{k} = \sum_{m=1}^{N} F_{ijm} \left[K^{-1}\right]^{mk} \tag{1.2.3}$$

which are symmetric in i, j and in general a-dependent. In terms of the matrices $[C_i]_j^k = C_{ij}^k$ the WDVV system expresses the commutation of C_i and C_j

$$[C_i, C_j] = 0 \tag{1.2.4}$$

Regarding the C_{ij}^k as structure constants of an algebra

$$\phi_i \phi_j = \sum_k C_{ij}^k \phi_k \tag{1.2.5}$$

we find that this algebra is commutative due to the symmetry in i, j and associative because of (1.2.4). Moreover, the algebra has a unit $\sum_q \alpha_q \phi_q$ where the α_q are the same as the ones occurring in the definition of K. Since this is true for all values of the a_i the structure constants in fact form a family of associative, commutative algebras with unit. We therefore have the following

Proposition 1.9. *A function $F(a_1, ..., a_N)$ satisfies the generalized WDVV equations if and only if*

- *There is a matrix $K = \sum_q \alpha_q F_q$ which is invertible but not necessarily constant*
- *there exists an N-parameter family of commutative, associative algebras with unit, whose structure constants C_{ij}^k are related to F through (1.2.3)*

It may seem that the generalized WDVV system depends on the particular linear combination K of third order derivatives, but this is not true. If the equations (1.2.2) hold for some K, they also hold for any other invertible linear combination of third order derivatives. In this sense the generalized WDVV equations are indeed a generalization of the original WDVV system, which puts additional conditions on K. To show this we need the following result [46]

Proposition 1.10. *If the generalized WDVV equations hold for a particular invertible linear combination K of third order derivatives, it holds simultaneously for all other invertible linear combinations.*

Proof. Assuming that the generalized WDVV equations hold with linear combination $K = \sum_q \alpha_q F_q$, we will prove that it also holds for $\tilde{K} = \sum_q \tilde{\alpha}_q F_q$ as long as $\tilde{K}^{-1}$ exists. Using $F_i = C_i K$ we write out

$$\begin{aligned} F_i \tilde{K}^{-1} F_j &= F_i \left[\sum_q \tilde{\alpha}_q F_q\right]^{-1} F_j = F_i \left[\sum_q \tilde{\alpha}_q C_q K\right]^{-1} F_j \\ &= F_i K^{-1} \left[\sum_q \tilde{\alpha}_q C_q\right]^{-1} F_j = C_i \left[\sum_q \tilde{\alpha}_q C_q\right]^{-1} C_j K \end{aligned} \tag{1.2.6}$$

The WDVV equations corresponding to K state that the C_i commute among each other and therefore $F_i\tilde{K}^{-1}F_j$ is symmetric in i and j. We conclude that the WDVV equations also hold for $\tilde{K}$. The relation between the structure constants D_{ij}^k corresponding with $\tilde{K}$ and C_{ij}^k corresponding with K are given by

$$C_i = F_i K^{-1} = F_i \tilde{K}^{-1}\tilde{K}K^{-1} = D_i \sum_q \tilde{\alpha}_q F_q K^{-1} = D_i \sum_q \tilde{\alpha}_q C_q \tag{1.2.7}$$

□

As a result, we can require the α_q occurring in K to be constant: replacing the a_i occurring in α_q by some constant values still leads to an invertible linear combination K. Note also that although the choice of α_q does not affect the function F, it does affect the structure constants C_{ij}^k of the family of associative and commutative algebras. Taking an algebra on a linear space V with basis elements $\{\phi_i\}$ and unit $e = \sum_q \alpha_q \phi_q$ the algebra reads

$$\phi_i\phi_j = \sum_k C_{ij}^k \phi_k = \sum_k C_{ij}^k \phi_k e \tag{1.2.8}$$

We can make a new algebra whose structure constants are given by

$$\phi_i\phi_j = \sum_k D_{ij}^k \phi_k \psi \tag{1.2.9}$$

For an invertible element $\psi = \sum_q \tilde{\alpha}_q \phi_q$ the D_{ij}^k are uniquely defined and form the structure constants of an associative commutative algebra. The relation between C_{ij}^k and D_{ij}^k is given by

$$\phi_i\phi_j = \sum_k C_{ij}^k \phi_k = \sum_l D_{ij}^l \phi_l \sum_q \tilde{\alpha}_q \phi_q = \sum_l D_{ij}^l \sum_q \tilde{\alpha}_q C_{ql}^k \phi_k \tag{1.2.10}$$

and corresponds precisely with a change of K from $\sum_q \alpha_q F_q$ to $\sum_q \tilde{\alpha}_q F_q$ as described in (1.2.7).

1.2.1 Physical background of the generalized WDVV equations and examples of solutions

The generalized WDVV system arose in the study of four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, also called Seiberg-Witten theory [59]. Although this theory is more complicated than two-dimensional superconformal field theory, Seiberg and Witten managed to solve the quantum low-energy behaviour exactly including the nonperturbative corrections. The solution is given in terms of a holomorphic function $\mathcal{F}(a_1, ..., a_N)$ which is called the prepotential[5]. This was a major breakthrough in quantum field theory, where results are often limited to perturbation theory and nonperturbative results are hard to come by.

After the initial work of Seiberg and Witten, prepotentials were given for $\mathcal{N} = 2$ supersymmetric Yang-Mills theories depending on various inputs, such as the dimension of space-time, the Lie algebra $\mathfrak{g}$ and the particle content of the theory. The main objects of study in this thesis are the prepotentials for four and five-dimensional spacetimes, for any simple Lie algebra

5 Here N denotes the rank of the Lie algebra $\mathfrak{g}$ of the gauge group G

and for a fixed particle content[6], and we will take some time here to sketch the physical background of the four-dimensional theory. A precise mathematical definition of $\mathcal{F}$ is postponed to chapters 2 and 3.

Quantum field theories typically give a good description of the point particles occurring in the theory in terms of perturbation theory, but they give a bad description of the solitonic objects occurring in it such as monopoles or instantons. The point particles are usually called local and the solitonic objects nonlocal. In the case of four-dimensional pure $\mathcal{N} = 2$ supersymmetric Yang-Mills theory the classical Lagrangian contains fields which describe the (local) gluons and their supersymmetric partners. As a first step towards quantization one finds a minimum of the potential and does perturbation theory around it. A typical feature of $\mathcal{N} = 2$ supersymmetric theories is that the classical potential has a whole family of minima, parametrized by a Weyl invariant polynomial u on the (complexified) Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Due to the Higgs effect, the gauge symmetry is broken from $\mathfrak{g}$ down to $\mathfrak{h}$ and the local particles split into two groups: on the one hand there are the massive particles and on the other there are the N massless local particles which are the gauge bosons for the gauge group $\mathfrak{h}$. There is a massive particle for each positive root α and its mass depends on u through a local function $a(u)$ taking values in the Cartan subalgebra:

$$M_\alpha = |(\alpha, a(u))| \tag{1.2.11}$$

Here $(.,.)$ denotes the Killing form.

This picture survives under quantization, i.e. there is still a family of quantum vacua and each vacuum defines its own physics since the mass spectrum depends on it. In a theory which is supposed to describe nature this is undesirable since there is no way to decide which vacuum is the 'real' one seen in nature. However, the existence of this space of vacua actually helps solve the low-energy physics. Since the theory has a mass gap it is expected that for low energies the theory can be described by a Lagrangian containing only the massless particles. The local function a is such that for generic large values of $|u|$ the masses M_α are big and perturbation theory is valid. In this regime it is found that the massless particles are all local and they are the gauge bosons. All massive particles are charged under the broken gauge group $\mathfrak{h}$. A generic massive particle is described by electrical charge numbers n_i^e and magnetic charge numbers n_i^m and is called a dyon. These charge numbers generate an element $q = \sum_{i=1}^N n_i^e \alpha_i$ and $g = \sum_{i=1}^N n_i^m \alpha_i^\vee$ in the root lattice $\Lambda_\mathfrak{g}$ and the coroot lattice $\Lambda_\mathfrak{g}^\vee$ respectively. The massive local particles are purely electrically charged and the solitonic objects acquire a magnetic charge. The mass of a dyon is given by

$$M_{g,q} = |(q, a) - (g, \frac{\partial \mathcal{F}(a)}{\partial a})| \tag{1.2.12}$$

Here the local holomorphic function $\mathcal{F}(a)$ is called the prepotential. The low-energy theory is given by an effective Lagrangian containing only fields corresponding to the massless local particles, and the information in this Lagrangian is equivalent to knowing $\mathcal{F}$.

This description is however not valid for all values of u. For certain values of u nonlocal particles can become massless, and they should be described in the effective Lagrangian of the low-energy theory. Seiberg and Witten have succesfully used the concept of duality in the solution of the low-energy behaviour: they suggest which solitonic objects can become

6 We consider only so-called pure Yang-Mills theory, i.e. only gluons and their supersymmetric partners and no quarks

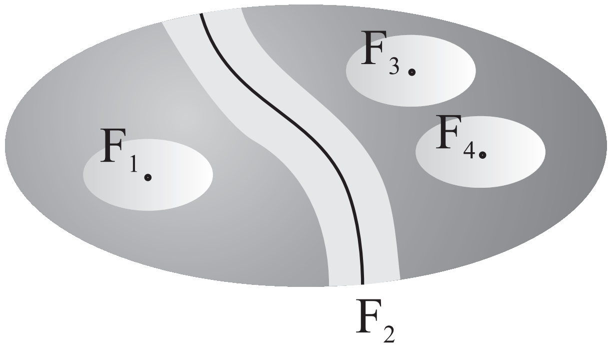

Figure 1.1: *Sketch of a moduli space with different singularities where dyons become massless. Around each singularity there is a patch of moduli space, each with its own prepotential describing the low-energy physics there. Picture taken from [43]*

massless and for which values of u, and they use a *different* quantum field theory which gives a good local description of that solitonic object in the neighbourhood of that point. The objects a and $\mathcal{F}$ can then change roles, a now describing masses of solitonic objects. With some further effort, it is shown that both $a_i = (\alpha_i, a)$ and $\frac{\partial \mathcal{F}}{\partial a_i}$ have monodromies when going around the points in the Cartan subalgebra where additional particles become massless. In fact, these points introduce singularities in $\mathbf{C}[\mathfrak{h}]^W$ where u takes its values, and we will denote the total subset of such points by Δ. The object $(a_i, \frac{\partial \mathcal{F}}{\partial a_i})$ turns out to be a section of a trivial vector bundle on $\mathbf{C}[\mathfrak{h}]^W - \Delta$ whose structure group is a subgroup of $Sp(2N, \mathbf{Z})$, the symplectic group. The mass formula (1.2.12) is invariant under these symplectic transformations. The identification of the structure group together with the local data around the singularities translate the low-energy description of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory into a Riemann-Hilbert problem.

Starting in the regime for large $|a_i|$ where the function $\mathcal{F}(a_1, ..., a_n)$ describes the low-energy physics, an element of the structure group will take us to a new $\tilde{a}_i$ regime, and since the structure group is symplectic we can integrate the corresponding object $\tilde{\mathcal{F}}_i$ to a function $\tilde{\mathcal{F}}(\tilde{a}_1, ..., \tilde{a}_N)$ which describes the physics in the new regime (see also section 1.2.4). All the local patches of $\mathbf{C}[\mathfrak{h}]^W - \Delta$ are therefore on equal footing, each having its own function $\mathcal{F}$ describing the low-energy physics, see also figure 1.1. In chapter 3 of this thesis we will show that all these functions $\mathcal{F}$ satisfy the generalized (but usually not the original!) WDVV equations. Indeed, the symplectic group $Sp(2N, \mathbf{Z})$ is shown in section 1.2.4 to be a group of symmetries of the generalized WDVV system and therefore all $\mathcal{F}$ satisfy the WDVV system if one of them does. Furthermore, these prepotentials can be written as a power series in an auxiliary parameter μ which serves as an energy scale in the physical theory. It is shown in chapter 2 that if the prepotential satisfies the WDVV equations, the zero order term in μ also does. This zero order term is called the perturbative prepotential, and as an example we give here the perturbative prepotential corresponding to the Lie algebra A_N.

Example 1.11. *For the gauge group $SU(N+1)$ with Lie algebra A_N, we give the four-dimensional perturbative prepotential*

$$\mathcal{F}_{pert} = \frac{1}{4} \sum_{i,j=1}^{N} (a_i - a_j)^2 \ln\left((a_i - a_j)^2\right) + \frac{1}{2} \sum_{i=1}^{N} a_i^2 \ln\left(a_i^2\right) \tag{1.2.13}$$

which satisfies the generalized WDVV equations (1.2.2). It can be checked that none of the matrices of third order derivatives is constant, and therefore $\mathcal{F}_{pert}$ does not satisfy the original WDVV system (1.1.2).

1.2.2 Mathematical background

The generalized WDVV system and corresponding Seiberg-Witten theory have made important contributions to various areas of mathematics. Here we touch upon some of these contributions, thus placing the system in its mathematical context.

The original WDVV equations are related to integrable systems in the sense that certain solutions to these equations are the logarithms of certain tau functions of the KdV hierarchy. Similarly, the generalized WDVV system is related to the Whitham dynamics of the periodic Toda chain since the Seiberg-Witten prepotentials are logarithms of tau functions of this integrable hierarchy [22], [48]. Although we do not discuss the Whitham hierarchy in this thesis, the periodic Toda chain and its relation with Seiberg-Witten theory is discussed briefly in section 3.1.2.

Another reason to study four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory is that its 'twisted' version[7] plays an important role in the definition of so-called Seiberg-Witten invariants of four-manifolds [63]. These invariants are equivalent to Donaldson's invariants but much simpler to calculate.

Finally, we note a connection between the generalized WDVV equations and the 'tau function of a curve' [61], [38]. In this context it is shown [9] that the logarithm of a certain tau function $\tau(t_0, t_1, \bar{t}_1, ..., t_N, \bar{t}_N)$ of the dispersionless 2D Toda hierarchy satisfies the generalized WDVV equations, when it is considered only as a function of the variables $t_0, t_1, ..., t_N$.

1.2.3 Integrable structure of the generalized WDVV equations

In section 1.1.3 it was noted that the original WDVV equations are equivalent to the compatibility conditions of the first order linear system (1.1.19) with spectral parameter z. In this section we prove a similar result, but without a spectral parameter, for the generalized WDVV system. Consider the first order linear system

$$(\partial_i + C_i D)\,\psi = 0 \qquad\qquad i = 1, ..., N \tag{1.2.14}$$

where $D = \sum_q \alpha_q \partial_q$ is a first order differential operator with constant coefficients, ψ is an N-dimensional vector of functions and the matrices $[C_i]^k_j$ in combination with D satisfy the following restrictions:

- The matrix $\sum_q \alpha_q C_q$ equals the $N \times N$ identity matrix
- There exists an invertible matrix K such that

  $$F_{ijm} := \sum_k C^k_{ij} K_{km} \tag{1.2.15}$$

  is totally symmetric in i, j, m.

7 A similar twist is made in the 2-dimensional supersymmetric conformal case

- The following relation holds

$$\partial_i K = D\left(C_i K\right) \tag{1.2.16}$$

We then have the following result (see also [54])

Proposition 1.12. *The compatibility conditions of the system (1.2.14) are equivalent with the generalized WDVV equations (1.2.2).*

Proof. The compatibility conditions are that $\partial_i \partial_j \psi = \partial_j \partial_i \psi$. Writing this out we get the following equation

$$\big(\partial_i C_j - \partial_j C_i - C_i D\left(C_j\right) + C_j D\left(C_i\right)\big) D + \left[C_i, C_j\right] D^2 = 0 \tag{1.2.17}$$

This is an operator identity, so the first and the second order term in D have to vanish separately. Writing out the first order term using the conditions (1.2.15) and (1.2.16) we find

$$\begin{aligned}
0 &= \partial_i C_j - \partial_j C_i + C_j D\left(C_i K K^{-1}\right) - C_i D\left(C_j K K^{-1}\right) \\
&= \partial_i C_j - \partial_j C_i + C_j\left(\partial_i K\right) K^{-1} - C_i\left(\partial_j K\right) K^{-1} + \left[C_j, C_i\right] K D\left(K^{-1}\right) \\
&= \left(\partial_i F_j - \partial_j F_i\right) K^{-1} + \left[C_j, C_i\right] K D\left(K^{-1}\right)
\end{aligned} \tag{1.2.18}$$

Therefore the compatibility conditions of (1.2.14) boil down to

$$\begin{aligned}
\left[C_i, C_j\right] &= 0 \\
\partial_i F_j - \partial_j F_i &= 0
\end{aligned} \tag{1.2.19}$$

and due to equation (1.2.15) the matrix K is identified as $K = \sum_q \alpha_q F_q$. The compatibility conditions are thus equivalent to the generalized WDVV system. ☐

The fact that the generalized WDVV system is equivalent to the compatibility conditions of a first order linear system is a sign that indicates that the system may be integrable, even though there is no spectral parameter in (1.2.14). This makes the system an interesting one to study.

1.2.4 Duality transformations and symmetries

In this section we consider two continuous symmetry groups of the generalized WDVV system: a general linear group of classical symmetries (consisting of linear changes of the variables), and a symplectic group of contact symmetries (consisting of so-called duality transformations). For the symmetries of the original WDVV equations, we refer to [18] and restrict ourselves to mentioning that the second symmetry group which we are about to discuss is not a symmetry group of the original system.

Lemma 1.13. *The generalized WDVV systems are invariant under linear changes of coordinates.*

Proof. We define a linear change of coordinates $\tilde{a}_i = \sum_j \left(A^{-1}\right)_i^j a_j$ and consider the new function

$$\tilde{F}(\tilde{a}_1, ..., \tilde{a}_N) = F(a_1, ..., a_N) \tag{1.2.20}$$

The third order derivatives of F transform under linear transformations as if they were the components of a $(3,0)$ tensor. From the definition of the structure constants of the associative and commutative algebra

$$C_{ij}^{k} = F_{ijl}\left(\left(\sum_{q}\alpha_q F_q\right)^{-1}\right)^{kl} \tag{1.2.21}$$

we find that the new objects

$$\tilde{C}_{ij}^{k} = \tilde{F}_{ijl}\left(\left(\sum_{q}\tilde{\alpha}_q \tilde{F}_q\right)^{-1}\right)^{kl} \tag{1.2.22}$$

are given by

$$\tilde{C}_{ij}^{k} = A_i^r A_j^s F_{rst}\left(\left(\sum_{q,s}\tilde{\alpha}_q A_q^u F_u\right)^{-1}\right)^{tv}\left(A^{-1}\right)_v^k \tag{1.2.23}$$

where the $\tilde{\alpha}_q$ are the transformed α_q. Here we have used the summation convention that indices occurring twice are summed over. The linear transformation is a symmetry if and only if the transformed objects $\tilde{C}_{ij}^k$ are the structure constants of an associative, commutative algebra with unit. To prove that this is indeed the case, we decompose the transformation of the C_{ij}^k into two steps. First we consider the objects

$$D_{ij}^{k} = F_{ijl}\left(\left(\sum_{q,s}\tilde{\alpha}_q A_q^s F_s\right)^{-1}\right)^{kl} \tag{1.2.24}$$

which are structure constants of an associative and commutative algebra with unit, see proposition 1.10. Let this algebra be defined on the linear space V with basis $\{\phi_i\}$. A linear change of coordinates in V then leads to the new structure constants $\tilde{C}_{ij}^k$ which are obtained from D_{ij}^k by considering it to be a (2,1) tensor, i.e.

$$\tilde{C}_{ij}^{k} = A_i^r A_j^s D_{rs}^t \left(A^{-1}\right)_t^k \tag{1.2.25}$$

This constitutes the second step in the transformation of the C_{ij}^k. The $\tilde{C}_{ij}^k$ are the structure constants of the same algebra as the D_{ij}^k (but with respect to a different basis in V) and the unit is given by $\sum_q \tilde{\alpha}_q \phi_q$. Thus the transformed function $\tilde{F}(\tilde{a}_1, ..., \tilde{a}_N)$ satisfies the generalized WDVV system. $\square$

We will now discuss a group of contact symmetries, which are different from classical symmetries in the sense that they do not only transform the variables a_i, but also the function F and its first order derivatives F_i. We recall that from physics it is known that a_i and F_i are to be treated on equal footing and together they form a section of a bundle with structure group $G \subset Sp(2N, \mathbf{Z})$. It is therefore tempting to suggest that $Sp(2N, \mathbf{Z})$ or a subgroup of it is a group of contact symmetries of the generalized WDVV system. Indeed, we have the following result [12], [14]

Proposition 1.14. *Let M be a constant $2N \times 2N$ matrix. The transformation*

$$\begin{pmatrix} \tilde{a}_1 \\ . \\ . \\ \tilde{a}_N \\ \tilde{F}_1 \\ . \\ . \\ \tilde{F}_N \end{pmatrix} = M \begin{pmatrix} a_1 \\ . \\ . \\ a_N \\ F_1 \\ . \\ . \\ F_N \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_1 \\ . \\ . \\ a_N \\ F_1 \\ . \\ . \\ F_N \end{pmatrix} \tag{1.2.26}$$

is a symmetry of the generalized WDVV system if and only if

- $\det(M) \neq 0$ *and* $M \in GL(1, \mathbf{C}) \times \mathbf{C} \otimes Sp(2N, \mathbf{R})$ *so* M *is up to a nonzero scalar given by an element of the complexified symplectic group,*

or

- $\det(M) = 0$ *and the third order derivatives of the transformed function are zero. We will specify precisely when this non interesting case occurs.*

Proof. For the $\tilde{a}_i$ to be a good set of coordinates we must have $\det\left(\frac{\partial \tilde{a}_i}{\partial a_j}\right) \neq 0$ and therefore

$$(A + BT)^{-1}$$

must exist, where T is the matrix given by $T_{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j}$. Since this condition depends on the initial function $F(a)$, we consider it to be a condition for generic F. Next we require a function $\tilde{F}(\tilde{a}_1, ..., \tilde{a}_N)$ to exist whose first order derivatives are the $\tilde{F}_i$, which is the same as demanding symmetricity of the matrix

$$\tilde{T} = (C + DT)(A + BT)^{-1} \tag{1.2.27}$$

Symmetricity of $\tilde{T}$ must hold for all T which come from a solution F of the WDVV equations, and in particular for any symmetric constant[8] matrix T. The symmetricity of $\tilde{T}$ therefore leads to the following three equations

$$\begin{aligned} A^T C &= C^T A \\ B^T D &= D^T B \\ \left[(B^T C - D^T A)T\right]^T &= T(B^T C - D^T A) \end{aligned} \tag{1.2.28}$$

Taking $T = I$ in the last of these equations implies that $B^T C - D^T A$ is a symmetric matrix, which again according to (1.2.28) should commute with all constant symmetric matrices T.

8 If T is constant, it originates from a second order polynomial F which is trivially a solution of the generalized WDVV system.

Therefore it is a multiple of the identity. This leads to the following three conditions

$$\begin{aligned} A^T C &= C^T A && (1.2.29) \\ B^T D &= D^T B && (1.2.30) \\ B^T C - D^T A &= \lambda I && (1.2.31) \end{aligned}$$

If $\lambda = 0$ then $\det(M) = 0$ and we will treat this case later. The effect of $\lambda \neq 0$ is that of a scaling on a_i and F_i with a factor $\sqrt{\lambda}$. Such a scaling is clearly a symmetry of the WDVV system, so we can divide it out and set $\lambda = -1$. Then $\det(M) = 1$ and the three conditions can be summarized in the single statement

$$M^T \Omega M = \Omega \tag{1.2.32}$$

where

$$\Omega = \begin{pmatrix} \emptyset & I \\ -I & \emptyset \end{pmatrix} \tag{1.2.33}$$

and therefore $M \in \mathbf{C} \otimes Sp(2N, \mathbf{R})$. Summarizing what we have done so far, we have seen that requiring the existence of a new function $\tilde{F}(\tilde{a})$ leads to the set of equations (1.2.29)-(1.2.31). We will find that the WDVV system (1.2.2) puts no further conditions on the matrix M. In order to show this, we will show that for nonzero λ the third order derivatives of F transform as the components of a $(3, 0)$ tensor, i.e.

$$\frac{\partial \tilde{F}}{\partial \tilde{a}_i \partial \tilde{a}_j \partial \tilde{a}_k} = \sum_{p,q,r} \frac{\partial a_p}{\partial \tilde{a}_i} \frac{\partial a_q}{\partial \tilde{a}_j} \frac{\partial a_r}{\partial \tilde{a}_k} \frac{\partial F}{\partial a_p \partial a_q \partial a_r} \tag{1.2.34}$$

This transformation is the same as for a linear change of coordinates apart from the fact that $\frac{\partial a_p}{\partial \tilde{a}_i}$ needn't be constant. The proof of lemma 1.13 can then be used to show that the transformation is a symmetry of the WDVV equations. In order to show that (1.2.34) indeed holds, we calculate

$$\frac{\partial \tilde{T}}{\partial a_k} = \left(D - \tilde{T} B\right) \frac{\partial T}{\partial a_k} (A + BT)^{-1} \tag{1.2.35}$$

Working out the first factor using (1.2.31) and symmetricity of $\tilde{T}$ we find

$$\begin{aligned} D - \tilde{T} B &= \left(A^T + TB^T\right)^{-1} \left(A^T + TB^T\right) \left(D - \tilde{T} B\right) \\ &= \left(A^T + TB^T\right)^{-1} \left(-\lambda I + C^T B + TD^T B - A^T \tilde{T} B - TB^T \tilde{T} B\right) \\ &= \lambda \left(A^T + TB^T\right)^{-1} && (1.2.36) \end{aligned}$$

and therefore writing out coefficients (1.2.35) becomes

$$\frac{\partial \tilde{T}_{ij}}{\partial a_k} = \lambda \sum_{p,q} \frac{\partial a_p}{\partial \tilde{a}_i} \frac{\partial T_{pq}}{\partial a_k} \frac{\partial a_q}{\partial \tilde{a}_j} \tag{1.2.37}$$

which for nonzero λ implies (1.2.34) up to a factor. For $\lambda = 0$ we find that the third order derivatives of $\tilde{F}$ are zero and this function trivially satisfies the WDVV system. The conditions (1.2.29)-(1.2.31) are therefore precisely the ones necessary to let the transformation (1.2.26) be a symmetry of the WDVV equations. For $\lambda = 0$ this symmetry will be trivial, for $\lambda \neq 0$ some highly nontrivial symmetries can occur. $\square$

Remark 1.15. *For transformations of the form (1.2.26) with* $\det(M) \neq 0$ *the condition that there should exist a new function* $\tilde{F}(\tilde{a})$ *already demands* M *to be symplectic up to a scalar. The generalized WDVV equations then put no further conditions on* M *whatsoever.*

To illustrate that the symplectic transformations just discussed can range from very simple to very complicated, we discuss two extremal situations, both playing an important role in physics. First consider a symplectic transformation of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & \emptyset \\ C & I \end{pmatrix} \tag{1.2.38}$$

Such transformations are called perturbative duality transformations and they only change the function F by adding quadratic pieces to it. These do not contribute to the third order derivatives and are obviously symmetries of the generalized WDVV system.

On the other hand, consider a transformation of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \emptyset & I \\ -I & \emptyset \end{pmatrix} \tag{1.2.39}$$

This transformation is called a nonperturbative or strong coupling duality transformation and it is known to be equal to a Legendre transform

$$\tilde{F}(\tilde{a}_1, ..., \tilde{a}_N) = F(a_1, ..., a_N) - \sum_i a_i F_i \tag{1.2.40}$$

Indeed, the matrix of coupling constants $T = \frac{\partial^2 F}{\partial a_i \partial a_j}$ is transformed to $-T^{-1}$ and the coupling constants are inverted. If the original coupling constants were large, the new ones will be small.

Since this Legendre transform is very complicated, we will restrict ourselves to a function of only one variable[9] to demonstrate what happens. Consider therefore a one-variable analogue of (1.2.13)

$$F(a) = \frac{1}{4} a^2 \log(a^2) - \frac{1}{4} a^2 \tag{1.2.41}$$

where the quadratic term has been inserted for later convenience. The transformed variable is given by

$$\tilde{a} = \frac{\partial F}{\partial a} = a \log(a) \tag{1.2.42}$$

This equation has infinitely many solutions for a, but only one of those equals 1 at $\tilde{a} = 0$. This solution is $a = \frac{\tilde{a}}{LW(\tilde{a})}$ where LW denotes Lambert's W function, which is analytic at zero. Using (1.2.40) we find that the transformed function equals

$$\tilde{F}(\tilde{a}) = -\frac{1}{4} \left(\frac{\tilde{a}}{LW(\tilde{a})} \right)^2 \log \left(\left(\frac{\tilde{a}}{LW(\tilde{a})} \right)^2 \right) - \frac{1}{4} \left(\frac{\tilde{a}}{LW(\tilde{a})} \right)^2 \tag{1.2.43}$$

9 A function of one variable trivially satisfies the WDVV system. The only purpose to consider it here is to show what the transformed function looks like.

1.3 Coordinate invariance and Frobenius manifolds

Roughly speaking, the WDVV equations express the fact that the third order derivatives of a function $F(t)$ form the structure constants of an associative and commutative algebra. This t-dependent family of algebras can be considered as a bundle over the space M where the t's live. It was Dubrovin's idea to identify this bundle with the tangent bundle TM of M and put the algebra structure on each of the tangent planes T_tM. This idea leads to the concept of a Frobenius manifold, a coordinate invariant description of the original WDVV equations. In this section we will define what a Frobenius manifold is and why it is not to be expected that a similar type of manifold can be introduced to describe the generalized WDVV system.

Definition 1.16. *[18] A Frobenius algebra is an associative, commutative algebra with a unit and a symmetric nondegenerate bilinear form $(.,.)$ on it such that*

$$(ab, c) = (a, bc) \tag{1.3.1}$$

Definition 1.17. *A Frobenius manifold M is a manifold with a Frobenius algebra structure on each of its tangent planes T_tM, depending smoothly on the point t and moreover satisfying:*

1. *$(.,.)$ is a flat metric on M*
2. *The unit vector field e is covariantly constant with respect to the Levi-Civita connection associated with the metric*

$$\nabla_u e = 0 \tag{1.3.2}$$

3. *The tensor (uv, w) is symmetric due to the Frobenius algebra structure. It can be differentiated and the result $\nabla_z(uv, w)$ is required to be symmetric in all vector fields u, v, w, z.*

Remark 1.18. *As we will see, in terms of certain special coordinates a Frobenius manifold leads to a solution F of the original WDVV equations. In the usual definition of a Frobenius manifold, there are additional requirements which lead to a certain quasi-homogeneity of the function F. To make the point of this section more clear, we have omitted these requirements.*

A solution F to the original WDVV system gives rise to a family of Frobenius algebras whose symmetric bilinear form $(.,.)$ is given by $(\partial_i, \partial_j) = F_{0ij}$. We mention the following result

Proposition 1.19. *[18] There exist coordinates t_i on any Frobenius manifold M such that the objects*

$$(\partial_i \partial_j, \partial_k) \tag{1.3.3}$$

are the third order derivatives of a function $F(t_0, ..., t_{N-1})$ satisfying the original WDVV equations. Moreover, all solutions to the original WDVV system are obtained in this way.

Proof. Due to the flatness condition 1, there exist coordinates t_i in terms of which the metric is constant. The covariant derivatives given through the Levi-Civita connection reduce to ordinary derivatives and therefore condition 3 ensures existence of a function F whose third order derivatives are given by

$$\frac{\partial^3 F(t)}{\partial t_i \partial t_j \partial t_k} = (\partial_i \partial_j, \partial_k) = \sum_l C^l_{ij} (\partial_l, \partial_k) \tag{1.3.4}$$

In terms of the unit $\sum_q \alpha_q \partial_q$ of the algebra, the metric is given by

$$(\partial_i, \partial_j) = \sum_q \alpha_q F_{qij} \tag{1.3.5}$$

and due to condition 2 the α_q are constant and we can make a linear change of coordinates (thus not losing constancy of the metric) in such a way that the metric is given by $\alpha_q = \delta_{0,q}$. To see that all solutions to the original WDVV equations are obtained in this way, one can easily construct a Frobenius manifold associated to each solution using the above formulas for the metric and its Levi-Civita connection. ☐

What we are interested in is the generalization of a Frobenius manifold to the setting of the generalized WDVV system. We should therefore drop all conditions in the definition of a Frobenius manifold which cause the metric to be given by the third order derivative of F with respect to a special coordinate t_0.

Definition 1.20. *A generalized Frobenius manifold is a manifold with a Frobenius algebra structure on each of its tangent planes T_tM, depending smoothly on the point t and moreover satisfying:*

1. $(.,.)$ *is a flat metric on* M
2. *The tensor* (uv, w) *is symmetric due to the Frobenius algebra structure. It can be differentiated and the result* $\nabla_z(uv, w)$ *is required to be symmetric in all vector fields* u, v, w, z.

The two conditions again ensure existence of a system of coordinates a_i such that there exists a function $F(a)$ whose third order derivatives are given by

$$\frac{\partial^3 F(a)}{\partial a_i \partial a_j \partial a_k} = (\partial_i \partial_j, \partial_k) = \sum_l C_{ij}^l (\partial_l, \partial_k) \tag{1.3.6}$$

In terms of the unit $\sum_q \alpha_q \partial_q$ of the algebra, the metric is given by

$$(\partial_i, \partial_j) = \sum_q \alpha_q F_{qij} \tag{1.3.7}$$

and equation (1.3.6) is equal to (1.2.3). Therefore the function F satisfies the generalized WDVV system. But although any generalized Frobenius manifold gives rise to a solution of the generalized WDVV equations, the converse statement is false because the linear combination $K = \sum_q \alpha_q F_q$ occurring in the generalized WDVV system is not required to be constant, whereas a generalized Frobenius manifold does have this condition due to the flatness of the metric. Of the three main types of solutions to the generalized WDVV system that we will study in this thesis, two in fact can be obtained from generalized Frobenius manifolds. These are the four- and five-dimensional perturbative prepotentials studied in chapter 2. For the other type of functions, studied in chapter 3, we do not have closed formulas and it is difficult to determine whether there exists a constant linear combination of third order derivatives. There is however nothing in the physical context nor in the construction of these functions that would suggest that such a combination exists [47].

We conclude that the generalized WDVV equations are probably not described by the most straightforward generalization of a Frobenius manifold, as introduced in definition 1.20. We will not pursue the coordinate invariant formulation of the generalized WDVV system any further.

Chapter 2

Perturbative prepotentials as explicit solutions

Chapter 2

This chapter deals with several perturbative prepotentials which are obtained in a certain limit from the full prepotentials of four and five-dimensional $\mathcal{N}=2$ supersymmetric Yang-Mills theory as discussed in section 2.1. We will restrict ourselves to pure Yang-Mills theories (describing gluons and not quarks) and find in section 2.2 that the perturbative prepotentials of the four-dimensional theory satisfy the generalized WDVV system. The five-dimensional prepotentials discussed in section 2.3 are more problematic, in the sense that they do not satisfy the WDVV equations. This problem can be overcome by introducing an extra variable a_0, which unexpectedly turns the prepotentials for all gauge groups into solutions of the original WDVV equations for the expanded set of variables, with a_0 playing the role of the special variable. Finally in section 2.4 we will discuss the role that certain physical parameters, viz. the energy scale and compactification radius, can play as new variables.

2.1 Perturbative limits

The solutions to the WDVV system coming from Seiberg-Witten theory typically depend on an energy scale μ. Despite its importance within the physical context, μ will play the role of an auxiliary parameter as far as the WDVV equations are concerned[1]. Whenever a solution to a system of linear differential equations can be written as a formal power series in a parameter μ, then the term of order zero in this expansion also satisfies this system. We will call such a zero order term a perturbative limit. We will now show explicitly that a perturbative limit of a solution of the nonlinear WDVV system also satisfies this system.

Suppose that the full solution and therefore the matrices of third order derivatives can be written as formal power series in a parameter μ

$$\begin{aligned} F(a_1,...,a_N) &= \sum_{p=0}^{\infty} F^p \mu^p = F^0(a_1,...,a_N) + F^1(a_1,...,a_N)\mu + ... \\ [F_k]_{lm} &= \sum_{p=0}^{\infty} [F_k^p]_{lm}\, \mu^p \end{aligned} \tag{2.1.1}$$

The inverse of a matrix of third order derivatives can also be expressed as a formal power

1 The possibility to treat μ as an additional variable for perturbative prepotentials will be considered in section 2.4

series by using the geometric series

$$\begin{aligned} [F_k]^{-1} &= \left(F_k^0\left(1+\left[F_k^0\right]^{-1}F_k^1\mu+...\right)\right)^{-1} \\ &= \sum_{q=0}^{\infty}\left(-\sum_{p=1}^{\infty}\left[F_k^0\right]^{-1}F_k^p\mu^p\right)^q\left[F_k^0\right]^{-1} \\ &= \left[F_k^0\right]^{-1}-\left[F_k^0\right]^{-1}F_k^1\left[F_k^0\right]^{-1}\mu+\mathcal{O}(\mu^2) \end{aligned} \tag{2.1.2}$$

Substituting these power series into the WDVV system (1.2.2), we find immediately that the perturbative limit F^0 satisfies the WDVV equations separately. This is not true in general for F^1, F^2 etcetera. As a counterexample, we will consider F^1 (the one-instanton correction) for the simplest case of a Seiberg-Witten prepotential, namely for four-dimensional pure gauge theory with A_N gauge group. In this case F^1 is given by (see e.g. [15])

$$F^1=\sum_{i=1}^{N}\frac{1}{a_i^2\prod_{j\neq i}(a_i-a_j)^2}+\prod_{j=1}^{N}\frac{1}{a_j^2} \tag{2.1.3}$$

We have checked explicitly for small N that it does not satisfy the WDVV system.

We have seen that perturbative limits of solutions to the WDVV equations are solutions themselves. Therefore if we can prove that the full prepotentials satisfy the system, we need not prove the same statement for their perturbative limits. There are however various reasons to study the perturbative limits in their own right. For one thing, they can be written down explicitly, in contrast to the full prepotentials. Furthermore, for the fact that the four-dimensional perturbative prepotentials satisfy the WDVV equations one can give a proof which makes the Lie algebraic background particularly clear. By studying this proof we can overcome the difficulties arising in the five-dimensional context.

We will now describe the perturbative limits of prepotentials for pure Seiberg-Witten theory. For any simple Lie algebra $\mathfrak{g}$ of rank N, consider the following function

$$F(a_1,...,a_N)=\sum_{\alpha\in R}f((\alpha,a)) \tag{2.1.4}$$

where $a=a_1e_1+...+a_Ne_N$ in terms of a basis $\{e_i\}$ of the root space R of $\mathfrak{g}$. The bracket $(.,.)$ represents the Killing form on the Cartan subalgebra of $\mathfrak{g}$. We will call f the base function of the prepotential F and the respective base functions for the four and five-dimensional physical theories are

$$f_4(x) = \frac{1}{2}x^2\log(x) \tag{2.1.5}$$

$$f_5(x) = \frac{1}{6}x^3-\frac{1}{4}Li_3(e^{-2x})=\frac{1}{6}x^3-\frac{1}{4}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^3} \tag{2.1.6}$$

In the process of proving that various prepotentials satisfy the WDVV system (1.2.2), it is very convenient to make a suitable choice for the linear combination K of third order derivatives of F. In the four-dimensional case we can take it to be the Killing form, whereas in the five-dimensional theory this is no longer possible. Other choices are then required to make K manageable, e.g. constant or diagonal.

The four-dimensional perturbative prepotentials are discussed in detail in section 2.2. Section 2.3 contains a discussion of the problems associated with five-dimensional prepotentials and finally section 2.3.2 resolves these problems in a natural way by adding an extra variable.

2.2 Four-dimensional perturbative prepotentials

All perturbative prepotentials of four-dimensional pure $\mathcal{N} = 2$ supersymmetric Yang-Mills theory are given by substituting into (2.1.4) the following base function

$$f_4(x) = \frac{1}{2}x^2 \log(x) \tag{2.2.1}$$

so that its third order derivative equals

$$f'''(x) = \frac{1}{x} \tag{2.2.2}$$

The Lie algebraic structure together with this particular f ensure that the function $F(a_1, ..., a_N)$ satisfies the WDVV system (1.2.2). As a matter of generalization, we can even take any root system associated to a Coxeter group to replace that of the Lie algebra.

Theorem 2.1. *[49] For any root system R of rank N, the function*

$$F(a_1, ..., a_N) = \frac{1}{2}\sum_{\alpha\in R}(\alpha, a)^2 \log((\alpha, a)) \tag{2.2.3}$$

satisfies the WDVV system. Here the bracket $(.,.)$ stands for the standard Euclidean inner product on the root space. In case R is the root system of a Lie algebra, this bracket equals the Killing form.

Proof. For three reasons we will adapt the proof in [49]. First, we no longer have to differentiate between long and short roots so that the proofs for simply laced and non simply laced Lie algebras become the same. Furthermore the adapted version allows the generalization to arbitrary root systems. Finally, the proof given below can easily be adapted to suit the five-dimensional situation.

The third order derivatives of F are given by

$$F_{ijk} = \sum_{\alpha\in R}\frac{1}{(\alpha, a)}(\alpha, e_i)(\alpha, e_j)(\alpha, e_k) \tag{2.2.4}$$

where we have taken a basis $\{e_1, ..., e_N\}$ for the root space. A natural choice for the matrix K is

$$K = \sum_{j=1}^{N} a_j F_j \tag{2.2.5}$$

where we recall that the notation F_j stands for a matrix of third order derivatives

$$(F_j)_{kl} = \frac{\partial^3 F}{\partial a_j \partial a_k \partial a_l} \tag{2.2.6}$$

Now the matrix K becomes

$$K_{kl} = \sum_{\alpha\in R}\frac{(\sum_j(\alpha, e_j)a_j)(\alpha, e_k)(\alpha, e_l)}{(\alpha, a)} = \sum_{\alpha\in R}(\alpha, e_k)(\alpha, e_l) \tag{2.2.7}$$

K is the matrix of a bilinear form on the root space R. Applying an element w of the Coxeter group on R, we find

$$K_{kl} \to \sum_{\alpha \in R} (\alpha, we_k)(\alpha, we_l) = \sum_{\alpha \in R} (w\alpha, e_k)(w\alpha, e_l) = K_{kl} \tag{2.2.8}$$

and since all bilinear forms which is invariant under the Coxeter group are proportional to the Euclidean metric, we conclude that K is the matrix of the Euclidean metric in the basis $\{e_i\}$. In the case of a simple Lie algebra, this is the matrix of the Killing form.

Taking $\{e_i\}$ to be an orthonormal basis[2], the left hand side of (1.2.2) becomes

$$\begin{aligned}\sum_{k,l} F_{ijk} \left(K^{-1}\right)_{kl} F_{lmn} - F_{mjk} \left(K^{-1}\right)_{kl} F_{lin} &= \sum_{k} F_{ijk}F_{kmn} - F_{mjk}F_{kin} \\ &= \sum_{\alpha,\beta \in R} \frac{(\alpha,\beta)(\alpha,e_j)(\beta,e_n)\left[(\alpha,e_i)(\beta,e_m) - (\alpha,e_m)(\beta,e_i)\right]}{(\alpha,a)(\beta,a)} \end{aligned} \tag{2.2.9}$$

Because this expression is antisymmetric in j and n, it is equal to

$$\begin{aligned}\frac{1}{2} \sum_{\alpha,\beta \in R} \frac{(\alpha,\beta)}{(\alpha,a)(\beta,a)} \left[(\alpha,e_i)(\beta,e_m) - (\alpha,e_m)(\beta,e_i)\right] \times \\ \left[(\alpha,e_j)(\beta,e_n) - (\alpha,e_n)(\beta,e_j)\right]\end{aligned} \tag{2.2.10}$$

In case $\alpha = \beta$, the contribution to (2.2.10) is zero. But if $\alpha \neq \beta$ the reflections in these two roots generate a nontrivial Weyl group element $\sigma_\alpha\sigma_\beta = w$. Thus we can split the sum in (2.2.10) into these Weyl group elements

$$\sum_{w \in W} \sum_{\substack{\sigma_\alpha\sigma_\beta = w \\ \alpha,\beta \in R}} \begin{aligned}\tfrac{(\alpha,\beta)}{(\alpha,a)(\beta,a)} \left[(\alpha,e_i)(\beta,e_m) - (\alpha,e_m)(\beta,e_i)\right] \times \\ \left[(\alpha,e_j)(\beta,e_n) - (\alpha,e_n)(\beta,e_j)\right]\end{aligned} \tag{2.2.11}$$

To prove that this expression equals zero, we will use the following Dunkl identity

Proposition 2.2. *[20] For any root system R with corresponding Coxeter group W we have*

$$\sum_{\substack{\sigma_\alpha\sigma_\beta = w \\ \alpha,\beta \in R}} B(\alpha,\beta)\frac{1}{(\alpha,a)(\beta,a)} = 0 \tag{2.2.12}$$

for any bilinear form B satisfying the following conditions

$$B(\alpha,\beta) = B(\beta,\alpha) \tag{2.2.13}$$

$$B(\sigma_\gamma\alpha,\sigma_\gamma\beta) = B(\alpha,\beta) \qquad \forall\gamma \in R \cap \{\mathbf{R}\alpha \oplus \mathbf{R}\beta\} \tag{2.2.14}$$

2 Changing the basis of the root space amounts to a linear change of the variables a_i, under which the WDVV system is invariant

In our situation we are dealing with

$$B(\alpha,\beta) = (\alpha,\beta)\,[(\alpha,e_i)(\beta,e_m) - (\alpha,e_m)(\beta,e_i)] \times \\ [(\alpha,e_j)(\beta,e_n) - (\alpha,e_n)(\beta,e_j)] \quad (2.2.15)$$

Clearly, condition (2.2.13) is satisfied. The most direct way to see that condition (2.2.14) also holds is to introduce the antisymmetric two-form C in the two-dimensional space $\mathbf{R}\alpha \oplus \mathbf{R}\beta$ by

$$C(x,y) = (x,e_i)(y,e_m) - (y,e_i)(x,e_m) \quad (2.2.16)$$

Since there is up to a constant only one antisymmetric bilinear form in a two-dimensional space, we find that under a reflection in $\mathbf{R}\alpha \oplus \mathbf{R}\beta$ the form $C(x,y)$ is only changed by a constant factor. Since a reflection has order two, the factor is ± 1. For sake of completeness, we will verify explicitly that both antisymmetric forms appearing in $B(\alpha,\beta)$ get the same factor -1 so that their product is invariant. Therefore condition (2.2.14) is satisfied.

We will use the following lemma:

Lemma 2.3. *For any root $\gamma \in \{\mathbf{R}\alpha \oplus \mathbf{R}\beta\}$ we have*

$$(\sigma_\gamma\alpha, e_i)(\sigma_\gamma\beta, e_m) = -(\alpha,e_i)(\beta,e_m) + \text{ symmetric} \quad (2.2.17)$$

where symmetric stands for terms which are symmetric in i and m.

Due to the antisymmetry of $B(\alpha,\beta)$ in i and m, the symmetric terms drop out and it is clear that this lemma ensures that condition (2.2.14) is met and therefore the WDVV system (1.2.2) is satisfied by the function F. We will now prove the lemma.

Proof. Writing out $(\sigma_\gamma\alpha, e_i)(\sigma_\gamma\beta, e_m)$, we find

$$(\sigma_\gamma\alpha, e_i)(\sigma_\gamma\beta, e_m) = (\alpha,e_i)(\beta,e_m) - \\ \frac{2}{(\gamma,\gamma)}\Big[(\alpha,\gamma)(\beta,e_m)(\gamma,e_i) + (\beta,\gamma)(\alpha,e_i)(\gamma,e_m)\Big] + \\ \frac{4}{(\gamma,\gamma)^2}(\alpha,\gamma)(\beta,\gamma)(\gamma,e_i)(\gamma,e_m) \quad (2.2.18)$$

where the last term is symmetric in i and m. Rewriting the rest using $\gamma = g_1\alpha + g_2\beta$, we find

$$(\alpha,e_i)(\beta,e_m) - \frac{2}{(\gamma,\gamma)}\Big[(\alpha,\gamma)(\beta,e_m)(\gamma,e_i) + (\beta,\gamma)(\alpha,e_i)(\gamma,e_m)\Big] \\ = (\alpha,e_i)(\beta,e_m) - \frac{2}{(\gamma,\gamma)}\Big[g_2(\alpha,\gamma)(\beta,e_m),(\beta,e_i) + g_1(\beta,\gamma)(\alpha,e_m),(\alpha,e_i) + \\ (g_1\alpha + g_2\beta,\gamma)(\alpha,e_i)(\beta,e_m)\Big] = -(\alpha,e_i)(\beta,e_m) + \ symmetric \quad (2.2.19)$$

□

□

This concludes the discussion of four-dimensional perturbative prepotentials. We will see in the next section that the five-dimensional situation is more complicated. The matrix K can no longer be taken to equal the matrix of the Killing form, and as a consequence of this we can no longer use the Dunkl identity. In fact, the generic five-dimensional perturbative prepotential does not satisfy the WDVV system. These problems and their resolutions will be discussed in the next section.

2.3 Five-dimensional perturbative prepotentials

The perturbative prepotentials of the five-dimensional gauge theories calculated with quantum field theory techniques are given by (2.1.4) with base function

$$f_5(x) = \frac{1}{6}x^3 - \frac{1}{4}Li_3(e^{-2x}) = \frac{1}{6}x^3 - \frac{1}{4}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^3} \tag{2.3.1}$$

whose third order derivative is

$$f_5'''(x) = \coth(x) \tag{2.3.2}$$

The corresponding prepotentials do *not* satisfy the WDVV system. The reason for this is that the conditions (2.2.13) and (2.2.14) are not satisfied and the Dunkl identity doesn't hold.

For type A_N Lie algebras it is shown in [24],[47] that the naive prepotential needs to be corrected by adding cubic polynomial terms coming from string theory. These terms are Weyl invariant and, as we will see later, they ensure that the type A prepotential now does satisfy the WDVV equations. Adding similar terms to the prepotentials for other classical Lie algebras leads only to partial success: first of all, there are no cubic Weyl invariant polynomials for Lie algebras of B, C, D type. Furthermore, adding such cubic terms despite the loss of Weyl invariance does not lead to solutions. If we include a free parameter in the prepotentials however, in such a way that the B and D type prepotentials are special cases, then we obtain solutions to the WDVV system. These solutions are given by fixing the new parameter to values which have no natural Lie algebraic interpretation. Therefore the connection with Lie algebras, which was so important in the four-dimensional case, seems to be lost for at least some of the classical gauge groups. The problems just described and their partial resolutions will be discussed in section 2.3.1.

The best results for the five-dimensional situation are obtained by adding an extra variable to the prepotentials. Remarkably, this simple procedure causes them to satisfy the original WDVV system and restores the important role of the Dunkl identity in the proof. These results will be discussed in section 2.3.2.

2.3.1 Problems in five dimensions

In this section we will give several theorems concerning five-dimensional prepotentials for classical Lie algebras, obtained in [28]. We will find that generically these prepotentials do not satisfy the WDVV system, in contrast to their four-dimensional counterparts.

We consider functions of the following type

$$
\begin{aligned}
F(a_1,...,a_N) \quad &= \sum_{1\le i<j\le N}\Big(\alpha_- f_5(a_i-a_j)+\alpha_+ f_5(a_i+a_j)\Big)+\eta\sum_{i=1}^{N} f_5(a_i) \\
&+ \quad \frac{a}{6}\left(\sum_{i=1}^{N} a_i\right)^3+\frac{b}{2}\left(\sum_{i=1}^{N} a_i\right)\left(\sum_{j=1}^{N} a_j^2\right)+\frac{c}{6}\sum_{i=1}^{N} a_i^3 \qquad (2.3.3)
\end{aligned}
$$

where we adopt the notation of [47]. The general form (2.3.3) is motivated by physics, see for instance [24],[4],[56]. In particular, the second line contains cubic terms coming from string theory, serving as corrections to the naive field theoretic perturbative prepotentials. These represent the most general cubic expression which is preserved by permutations of the variables $a_1,...,a_N$. The perturbative prepotentials for A, B and D type Lie algebras are obtained as special cases of this general function F.

For various combinations of the parameters we will investigate whether or not F satisfies the WDVV system (1.2.2). The method used involves making an appropriate choice for the matrix K, although the results are of course independent of this particular choice.

2.3.1.1 The simplest case

The simplest set of parameters we consider is $\alpha_+ = \eta = 0$. These values do not correspond to an actual prepotential from physics, but we do find solutions to the WDVV system. Without loss of generality we can chose $\alpha_- = 1$ by scaling a, b, c.

We can prove the following result

Theorem 2.4. *The function (2.3.3) with $\alpha_- = 1$, $\alpha_+ = 0$ and $\eta = 0$ satisfies the WDVV system (1.2.2) if and only if the following relation holds*

$$
Nb^3 + 3b^2c - ac^2 + 3Nb + c + N^2a = 0 \qquad (2.3.4)
$$

More accurately, this relation is correct in the generic case that both $Nb + c \neq 0$ and $Na + 2b \neq 0$. Special cases will be discussed separately in the proof.

Proof. Writing $\beta_{ij} = f'''(a_i - a_j)$, the third order derivatives of F are

$$
\begin{aligned}
F_{klm} = a + \delta_{kl}\delta_{lm}\left(\sum_{q\neq k}\beta_{kq} + 3b + c\right) + \delta_{kl}(1-\delta_{km})(\beta_{mk}+b) + \\
\delta_{km}(1-\delta_{kl})(\beta_{lk}+b) + \delta_{lm}(1-\delta_{kl})(\beta_{kl}+b) = aU_{lm} + (V_k)_{lm} \quad (2.3.5)
\end{aligned}
$$

We take a specific linear combination $K = \sum_{j=1}^{N} F_j$ and using (2.3.2) we find

$$
K = (Na + 2b)U + (Nb + c)I \qquad (2.3.6)
$$

Special situations occur when $Na + 2b = 0$ and / or $Nb + c = 0$. The first results in K being a multiple of the identity and the second causes K to become singular. For the moment we will work with generic K and we will come back to the special cases later. The inverse of K equals up to a factor

$$K^{-1}_{kl} = 1 + \delta_{kl}\left(-\frac{Nb+c}{Na+2b} - N\right) \tag{2.3.7}$$

For the WDVV equations to hold, we should have

$$\left(F_i K^{-1} F_m\right)_{jn} - \left(F_m K^{-1} F_i\right)_{jn} = 0 \tag{2.3.8}$$

or equivalently

$$K_{ij}K_{mn} - K_{mj}K_{in} - \frac{3Nb + c + N^2 a}{Na+2b}\left[F_i, F_m\right]_{jn} = 0 \tag{2.3.9}$$

We will first calculate the commutator

$$[F_i, F_m] = [aU + V_i, aU + V_m] \tag{2.3.10}$$

We find

$$(UV_m)_{kl} = 2b + \delta_{lm}(1-\delta_{kl})(Nb+c) \tag{2.3.11}$$

and since $U^T = U$ and $V_m^T = V_m$ we also know $V_m U = (UV_m)^T$. Furthermore, if we use the identity

$$\beta_{ij}\beta_{ik} + \beta_{ij}\beta_{kj} + \beta_{ik}\beta_{jk} = 1 \tag{2.3.12}$$

we find

$$\begin{aligned}
[V_i, V_m]_{jn} &= \delta_{ij}(1-\delta_{mn})(1-\delta_{in})(b^2-1) + \delta_{mn}(1-\delta_{jm})(1-\delta_{ij})(b^2-1)\\
&- \delta_{jm}(1-\delta_{mn})(1-\delta_{in})(b^2-1) - \delta_{in}(1-\delta_{jm})(1-\delta_{ij})(b^2-1)\\
&+ \delta_{ij}\delta_{mn}(\beta + 2(b^2-1)) - \delta_{jm}\delta_{in}(\beta + 2(b^2-1))
\end{aligned} \tag{2.3.13}$$

and therefore

$$\begin{aligned}
[F_i, F_m]_{jn} = \delta_{ij}(1-\delta_{mn})(1-\delta_{in})\alpha + \delta_{mn}(1-\delta_{jm})(1-\delta_{ij})\alpha -\\
\delta_{jm}(1-\delta_{mn})(1-\delta_{in})\alpha - \delta_{in}(1-\delta_{jm})(1-\delta_{ij})\alpha +\\
\delta_{ij}\delta_{mn}(\beta - 2\alpha) - \delta_{jm}\delta_{in}(\beta - 2\alpha)
\end{aligned} \tag{2.3.14}$$

where

$$\begin{aligned}
\alpha &= b^2 - 1 - ac - Nab\\
\beta &= N + Nb^2 + 2bc
\end{aligned} \tag{2.3.15}$$

On the other hand, we have

$$\begin{aligned}
K_{ij}K_{mn} - K_{mj}K_{in} = (Na+2b)^2\Big[&\delta_{ij}(1-\delta_{mn})(1-\delta_{in})\gamma +\\
&\delta_{mn}(1-\delta_{jm})(1-\delta_{ij})\gamma - \delta_{jm}(1-\delta_{mn})(1-\delta_{in})\gamma\\
&- \delta_{in}(1-\delta_{jm})(1-\delta_{ij})\gamma + \delta_{ij}\delta_{mn}(\delta - 2\gamma) - \delta_{jm}\delta_{in}(\delta - 2\gamma)\Big]
\end{aligned} \tag{2.3.16}$$

where

$$\gamma = \frac{Nb + c}{Na + 2b} \qquad\qquad \delta = \gamma^2 \tag{2.3.17}$$

The equation (2.3.9) therefore reduces to two algebraic relations among the parameters a, b, c. These relations are

$$-\frac{3Nb + c + N^2 a}{Na + 2b}\alpha + (Na + 2b)^2\gamma = 0 \tag{2.3.18}$$

and

$$-\frac{3Nb + c + N^2 a}{Na + 2b}(2\alpha - \beta) + (Na + 2b)^2(2\gamma - \delta) = 0 \tag{2.3.19}$$

which combine into only one relation

$$Nb^3 + 3b^2c - ac^2 + 3Nb + c + N^2 a = 0 \tag{2.3.20}$$

This finishes the proof of theorem 2.4 for the generic case where both $Nb + c \neq 0$ and $Na + 2b \neq 0$.

There are special situations for either $Na + 2b = 0$ or $Nb + c = 0$ or both. If $Na + 2b = 0$ and $Nb + c \neq 0$ then we find that the WDVV equations hold if and only if

$$[F_i, F_m] = 0 \tag{2.3.21}$$

and therefore if and only if

$$\alpha = 1 + \left(\frac{Na}{2}\right)^2 - ac = 0$$

and

$$2\alpha - \beta = -(N - 2)\left(1 + \left(\frac{Na}{2}\right)^2 - ac\right) = 0 \tag{2.3.22}$$

Note that just substituting $b = -\frac{Na}{2}$ in (2.3.20) gives

$$(N^2 a - 2c)\left(1 + \left(\frac{Na}{2}\right)^2 - ac\right) = 0 \tag{2.3.23}$$

which is only partially correct since $N^2 a - 2c = 0$ does not yield a solution.

Furthermore, if $Nb + c = 0$ and $Na + 2b \neq 0$, then (2.3.6) shows that K becomes singular. Experience tells us that for $N \neq 3$ there exist no solutions to the WDVV equations without the extra requirement $b = \pm 1$. For $N = 3$ there is no such condition on b and the WDVV equations are satisfied. We will now consider $b = 1$ and $b = -1$ separately. If $b = 1$ then we chose a new nonsingular K equal to

$$K = \sum_{j=1}^{N} h_j F_j = \sum_{j=1}^{N}\left(-(2 + a(N - 1))e^{2a_j} + a\sum_{i \neq j} e^{2a_i}\right)F_j \tag{2.3.24}$$

and working this out we find that K equals up to a factor

$$\left(\frac{Na}{2}+b\right)I \tag{2.3.25}$$

which is a nonzero multiple of the identity since $Na+2b\neq 0$. If $b=-1$ on the other hand, we take

$$K=\sum_{j=1}^{N}h_jF_j=\sum_{j=1}^{N}\left(\prod_{k\neq j}(2-a(N-1))e^{2a_k}+a\sum_{k\neq j}\prod_{i\neq k}e^{2a_i}\right)F_j \tag{2.3.26}$$

which also leads to K being a multiple of the identity. So in both cases we must solve (2.3.21) again, which leads to

$$Na+2b=0 \tag{2.3.27}$$

which is precisely what we excluded before.

Finally, if we take both $Na+2b=0$ and $Nb+c=0$ then all linear combinations of the F_j become singular and the WDVV equations are meaningless.

Summarizing, we conclude that if $Na+2b=0$ and $Nb+c\neq 0$ there are solutions if and only if

$$1+\left(\frac{Na}{2}\right)^2-ac=0 \tag{2.3.28}$$

and if $Nb+c=0$ and $Na+2b\neq 0$ there are solutions if and only if $N=3$ and finally if both $Na+2b=0$ and $Nb+c=0$ then there are no solutions at all. This finishes the discussion of theorem 2.4. $\square$

2.3.1.2 The type A prepotential

Let us now turn to a prepotential with physical background. We consider the function

$$\tilde{F}(x_1,...,x_{N+1})=\sum_{1\leq i<j\leq N+1}f_5(x_i-x_j)+\frac{N+1}{2}\sum_{1\leq i<j<k\leq N+1}x_ix_jx_k \tag{2.3.29}$$

which is of the form (2.3.3) with parameters a,b,c given by

$$a=\frac{N+1}{2}\qquad b=-\frac{N+1}{2}\qquad c=N+1 \tag{2.3.30}$$

The $SU(N+1)$ perturbative prepotential is obtained from $\tilde{F}$ by the linear [3] change of variables

$$\begin{aligned} a_i &= x_i-x_{N+1} && i=1,...,N \\ a_{N+1} &= x_1+...+x_{N+1} \end{aligned} \tag{2.3.31}$$

3 The WDVV equations are invariant under linear coordinate changes

and the substitution $a_{N+1} = 0$. Concretely it is given by

$$F(a_1, ..., a_N) = \sum_{1\le i<j\le N} f_5(a_i - a_j) + \sum_{i=1}^{N} f_5(a_i) + \frac{1}{3(N+1)} \left(\sum_{i=1}^{N} a_i\right)^3 - \frac{1}{2}\left(\sum_{i=1}^{N} a_i\right)\left(\sum_{j=1}^{N} a_j^2\right) + \frac{N+1}{6}\sum_{i=1}^{N} a_i^3 \quad (2.3.32)$$

This is of the general type (2.3.3) with parameters

$\alpha_- = 1$	$\alpha_+ = 0$	$\eta = 1$
$a = \frac{2}{N+1}$	$b = -1$	$c = N+1$

It turns out that the sign of the correction term in (2.3.29) is irrelevant for the WDVV equations.

We can confirm the result in [47] and prove

Theorem 2.5. *The function*

$$\begin{aligned} F(a_1, ..., a_N) &= \sum_{1\le i<j\le N} f_5(a_i - a_j) + \sum_{i=1}^{N} f_5(a_i) \pm \frac{1}{3(N+1)} \left(\sum_{i=1}^{N} a_i\right)^3 \\ &\mp \frac{1}{2}\left(\sum_{i=1}^{N} a_i\right)\left(\sum_{j=1}^{N} a_j^2\right) \pm \frac{N+1}{6}\sum_{i=1}^{N} a_i^3 \end{aligned} \quad (2.3.33)$$

satisfies the WDVV system (1.2.2).

Remark 2.6. *We note that (2.3.33) is invariant under the Weyl group of A_N. In fact, taking arbitrary values for a, b, c this is still the case. A natural question is therefore whether an F with $\alpha_- = 1$, $\alpha_+ = 0$ and $\eta = 1$ satisfies the WDVV system for any other values of a, b, c. Calculations for ranks up to five suggest that there are no other solutions. In particular, the naive prepotential with $a, b, c = 0$ coming from quantum field theory does not satisfy the WDVV system. This should mean that the string theory corrections are* precisely *the ones needed to satisfy the WDVV equations.*

Proof. Taking $\alpha_- = 1, \alpha_+ = 0, \eta = 1$ and a, b, c arbitrary F has third order derivatives equal to

$$F_{klm} = a + \delta_{kl}\delta_{lm}M_k + \delta_{kl}\beta_{mk} + \delta_{km}\beta_{lk} + \delta_{lm}\beta_{kl} \quad (2.3.34)$$

where

$$\begin{aligned} M_k &= \sum_{q\neq k} \beta_{kq} + \beta_k \\ \beta_k &= \eta \coth(a_k) + (4-N)b + c \\ \beta_{ij} &= \begin{cases} 0 & \text{if} \quad i = j \\ \alpha_- \coth(a_i - a_j) + \alpha_+ \coth(a_i + a_j) + b & \text{if} \quad i \neq j \end{cases} \end{aligned} \quad (2.3.35)$$

Consider a linear combination K of the following form

$$K_{kl} = \delta_{kl}\frac{1}{A_k} \tag{2.3.36}$$

where A_k depends on the specific prepotential under consideration and will be specified later. We find

$$\begin{aligned}
\left(F_i K^{-1} F_m\right)_{jl} &= \sum_{k=1}^{N} F_{ijk} A_k F_{klm} \\
&= \delta_{im}\Big(A_i\beta_{ji}\beta_{lm} + \delta_{ij}\beta_{lm}A_iM_i + \delta_{il}\beta_{ji}A_iM_i + \delta_{ij}\delta_{lm}A_iM_iM_m\Big) \\
&+ \delta_{jl}(1-\delta_{lm})A_j\beta_{ij}\beta_{mj} \\
&+ \delta_{jl}\delta_{lm}A_mM_l\beta_{im} \\
&+ \delta_{ij}\delta_{il}A_iM_l\beta_{mi} \\
&+ \delta_{il}(1-\delta_{jm})A_i\beta_{ji}\beta_{mi} \\
&+ \delta_{lm}(1-\delta_{ij})\left(A_i\beta_{ji}\beta_{im} + A_j\beta_{ij}\beta_{jm} + aA_mM_m + a\sum_k A_k\beta_{km}\right) \\
&+ \delta_{jm}(1-\delta_{il})A_j\beta_{ij}\beta_{lj} \\
&+ \delta_{ij}(1-\delta_{lm})\left(A_l\beta_{li}\beta_{ml} + A_m\beta_{mi}\beta_{lm}aA_iM_i + a\sum_k A_k\beta_{ki}\right) \\
&+ \delta_{ij}\delta_{lm}\Bigg(A_iM_i\beta_{im} + A_mM_m\beta_{mi} + \sum_{k\neq i,m} A_k\beta_{km}\beta_{ki} + aA_mM_m \\
&\qquad\qquad + a\sum_k A_k\beta_{km} + aA_iM_i + a\sum_k A_k\beta_{ki}\Bigg) \\
&+ \delta_{jm}\delta_{il}\left(A_m\beta_{im}^2 + A_i\beta_{mi}^2\right) \\
&+ a^2\sum_k A_k + a(A_j\beta_{ij} + A_i\beta_{ji}) + a(A_m\beta_{lm} + A_l\beta_{ml})
\end{aligned} \tag{2.3.37}$$

Here it should be noted that the last line contributes to all the previous ones. For example, if $i = l, i \neq m, i \neq j, j \neq m$ then $\left(F_i K^{-1} F_m\right)_{jl}$ is not

$$A_j\beta_{ij}\beta_{mj} \tag{2.3.38}$$

but rather

$$A_j\beta_{ij}\beta_{mj} + a^2\sum_k A_k + a(A_j\beta_{ij} + A_i\beta_{ji}) + a(A_m\beta_{im} + A_i\beta_{mi}) \tag{2.3.39}$$

In order to satisfy the WDVV system we should check whether or not (2.3.37) is symmetric in i and m. For example, the first two lines are automatically preserved under the interchange of i and m. The third and fourth lines on the other hand are mutually exchanged. The rest of condition (1.2.2) is nontrivial and depends on the details of the function F.

The two cases $a = \pm\frac{2}{N+1}$, $b = \mp 1$, $c = \pm(N+1)$ have to be treated separately.

The case $\mathbf{a = -\frac{2}{N+1}}$, $\mathbf{b = 1}$, $\mathbf{c = -(N+1)}$:
For this case, we take a specific linear combination $K = \sum_j h_j F_j$ where

$$h_j = e^{2a_j} + \sum_{i=1}^{N} e^{2a_i} \tag{2.3.40}$$

and we find up to a factor

$$K_{kl} = \delta_{kl}\left(\frac{1}{1 - e^{-2a_k}}\right) = \delta_{kl}\frac{1}{A_k} \tag{2.3.41}$$

Using this information we can derive the following identities

$$A_j\beta_{ij} + A_i\beta_{ji} = 2 - \frac{2}{e^{2a_i}} - \frac{2}{e^{2a_j}} \tag{2.3.42}$$

$$A_i\beta_{ij} + A_j\beta_{ji} = 2 \tag{2.3.43}$$

$$A_i\beta_{ji}\beta_{im} + A_j\beta_{ij}\beta_{jm} - A_m\beta_{jm}\beta_{im} = \frac{4}{e^{2a_m}} \tag{2.3.44}$$

Turning to the WDVV condition we find that the first two lines of (2.3.37) are preserved under the interchange of i and m and that the third and fourth lines become mutually exchanged. We will now study the fifth and sixth lines. Keeping in mind that the last line of (2.3.37) contributes to both of these, we find that the fifth line becomes

$$\delta_{il}(1-\delta_{lm})\Big(A_j\beta_{ij}\beta_{mj} + a^2\sum_k A_k + a\,(A_j\beta_{ij} + A_i\beta_{ji}) + a\,(A_m\beta_{lm} + A_l\beta_{ml})\Big)$$

and the sixth becomes

$$\delta_{lm}(1-\delta_{ij})\Big(A_i\beta_{ji}\beta_{im} + A_j\beta_{ij}\beta_{jm} + a^2\sum_k A_k + a\,(A_j\beta_{ij} + A_i\beta_{ji}) + a\sum_k A_k\beta_{km} + aA_mM_m\Big)$$

Using the definition of M_m and the relations (2.3.42), (2.3.43) and (2.3.44) we see that these are indeed exchanged under the interchange of i and m. The seventh and eighth lines of (2.3.37) are mutually exchanged for the same reasons, which leaves us with the ninth and tenth lines. The complete ninth line becomes

$$\delta_{ij}\delta_{lm}\Big(A_iM_i\beta_{im} + A_mM_m\beta_{mi} + \sum_{k\neq i,m} A_k\beta_{km}\beta_{ki} + aA_mM_m + a\sum_k A_k\beta_{km} + aA_iM_i + a\sum_k A_k\beta_{ki} + a^2\sum_k A_k\Big) \tag{2.3.45}$$

and the tenth line is

$$\delta_{jm}\delta_{il}\left(A_m\beta_{im}^2 + A_i\beta_{mi}^2 + 2a\,(A_m\beta_{im} + A_i\beta_{mi}) + a^2\sum_k A_k\right) \tag{2.3.46}$$

Using the definition of M_m and working out (2.3.45) we find

$$\delta_{ij}\delta_{lm}\Bigg(\sum_{k\neq i,m}(A_k\beta_{ik}\beta_{im}+A_m\beta_{mk}\beta_{mi}+A_k\beta_{km}\beta_{ki})+A_i\beta_{im}^2+ \\ A_m\beta_{mi}^2+A_i\beta_i\beta_{im}+A_m\beta_m\beta_{mi}+a\sum_{k\neq m}(A_m\beta_{mk}+A_k\beta_{km})+ \\ aA_m\beta_m+a\sum_{k\neq i}(A_i\beta_{ik}A_k\beta_{ki})+aA_i\beta_i+a^2\sum_k A_k\Bigg) \tag{2.3.47}$$

We will make use of (2.3.43) and the following relations

$$A_i\beta_{ik}\beta_{im}+A_m\beta_{mk}\beta_{mi}+A_k\beta_{km}\beta_{ki} = 4 \tag{2.3.48}$$

$$A_i\beta_i\beta_{im}+A_m\beta_m\beta_{mi} = 8-4N \tag{2.3.49}$$

and we find that the ninth line becomes

$$\delta_{ij}\delta_{lm}\left(A_i\beta_{im}^2+A_m\beta_{mi}^2+4a(N-1)+a\left(A_m\beta_m+A_i\beta_i\right)+a^2\sum_k A_k\right) \tag{2.3.50}$$

Using (2.3.43) again we find that the tenth line becomes

$$\delta_{jm}\delta_{il}\left(A_m\beta_{im}^2+A_i\beta_{mi}^2+4a-\frac{4a}{e^{2a_i}}-\frac{4a}{e^{2a_m}}+a^2\sum_k A_k\right) \tag{2.3.51}$$

and using the relations

$$A_i\beta_{im}^2+A_m\beta_{mi}^2-A_m\beta_{im}^2-A_i\beta_{mi}^2 = \frac{4}{e^{2a_i}}+\frac{4}{e^{2a_m}} \tag{2.3.52}$$

$$8-4N+\frac{2N-2}{e^{2a_i}}+\frac{2N-2}{e^{2a_m}} = A_m\beta_m+A_i\beta_i \tag{2.3.53}$$

we find that the ninth and tenth lines are indeed exchanged under the interchange of i and m. Therefore the prepotential (2.3.33) satisfies the WDVV equations and we have proven half of theorem 2.5.

The case $\mathbf{a}=\frac{\mathbf{2}}{\mathbf{N+1}}$, $\mathbf{b}=-\mathbf{1}$, $\mathbf{c}=(\mathbf{N}+\mathbf{1})$:

In this case, taking

$$h_j=e^{2a_j}+\sum_{i=1}^{N}e^{2a_i} \tag{2.3.54}$$

we find that up to a factor $K=\sum_j h_jF_j$ equals

$$\delta_{kl}\left(\frac{1}{1-e^{2a_k}}\right)=\delta_{kl}\frac{1}{A_k} \tag{2.3.55}$$

So with respect to the previous case there are modifications in the definitions of β_k, β_{ij} and A_k. This causes the relations (2.3.42), (2.3.43), (2.3.44), (2.3.48), (2.3.49), (2.3.52) and (2.3.53) to be changed to the following ones

$$\begin{aligned}
A_j\beta_{ij} + A_i\beta_{ji} &= -2 + 2e^{2a_i} + 2e^{2a_j} \\
A_i\beta_{ij} + A_j\beta_{ji} &= -2 \\
A_i\beta_{ji}\beta_{im} + A_j\beta_{ij}\beta_{jm} - A_m\beta_{jm}\beta_{im} &= 4e^{2a_m} \\
A_i\beta_{ik}\beta_{im} + A_m\beta_{mk}\beta_{mi} + A_k\beta_{km}\beta_{ki} &= 4 \\
A_i\beta_i\beta_{im} + A_m\beta_m\beta_{mi} &= 4N - 8 \\
A_i\beta_{im}^2 + A_m\beta_{mi}^2 - A_m\beta_{im}^2 - A_i\beta_{mi}^2 &= 4e^{2a_i} + 4e^{2a_m} \\
A_m\beta_m + A_i\beta_i &= 4N - 8 - \\
&\quad (2N-2)e^{2a_i} - (2N-2)e^{2a_m}
\end{aligned}$$

and using these relations we find that the WDVV equations are again satisfied. This proves theorem 2.5. □

2.3.1.3 Other classical Lie algebras

Next we consider a prepotential inspired by other classical Lie algebras. Without correction terms, the B, D prepotentials are both given by $\alpha_- = 1$, $\alpha_+ = 1$ and by $\eta = 1, 0$ respectively. Leaving the parameter η unfixed, we can prove the following theorem

Theorem 2.7. *The function*

$$F(a_1, ..., a_N) = \sum_{1\leq i<j\leq N} \Big(f_5(a_i - a_j) + f_5(a_i + a_j) \Big) + \eta \sum_{i=1}^{N} f_5(a_i) \qquad (2.3.56)$$

satisfies the WDVV equations (1.2.2) if and only if $\eta = -2(N-2)$.

Remark 2.8. *This solution seems to have little to do with the B, D Lie algebras, since η must take on a fixed special value different from 0 and 1. One can think about restoring the Lie algebraic interpretation by adding third order correction terms. It is conceivable that their contribution gives enough freedom to take η equal to 0 or 1. But the Lie algebras under consideration do not possess any third order Weyl invariant polynomials and therefore the correction terms would automatically spoil Weyl invariance and any Lie algebraic interpretation with it. Fourth (or higher) order correction terms should therefore be used which give nonconstant additions to the third order derivatives of F, which is beyond the scope of this thesis.*

As an alternative, we can introduce an extra auxiliary variable which can be multiplied by the second *order Weyl invariant polynomial that all simple Lie algebras have. The success of this approach is remarkable and will be discussed in detail in section 2.3.2. For the sake of completion we mention that calculations for low ranks indicate that adding third order correction terms with parameters a, b, c indeed doesn't help: the WDVV equations always force $a = b = c = 0$.*

Proof. The first part of the proof is identical to the first part of the proof of theorem 2.5. The third order derivatives are given by (2.3.34) with

$$\begin{aligned}
M_k &= \sum_{q\neq k}\beta_{kq}+\beta_k \\
\beta_k &= \eta\coth(a_k) \\
\beta_{ij} &= \begin{cases} 0 & \text{if} \quad i=j \\ \coth(a_i-a_j)+\coth(a_i+a_j) & \text{if} \quad i\neq j \end{cases} \\
a &= b = c = 0
\end{aligned} \tag{2.3.57}$$

One can derive the following relations

$$\beta_{ji}\beta_{im}+\beta_{ij}\beta_{jm}-\beta_{jm}\beta_{im} = 0 \tag{2.3.58}$$
$$\beta_{ik}\beta_{im}+\beta_{mk}\beta_{mi}+\beta_{km}\beta_{ki} = 4 \tag{2.3.59}$$
$$\beta_i\beta_{im}+\beta_m\beta_{mi} = 2 \tag{2.3.60}$$

which are identities that we will need later. Furthermore, we take

$$K_{kl}=\sum_{j=1}^{N}\sinh(2a_j)F_{jkl} \tag{2.3.61}$$

and using (2.3.58) we find

$$K_{kl}=\delta_{kl}\left(1-N+\sum_{j=1}^{N}\cosh^2(a_j)+\frac{1}{2}\left(2(N-2)+\eta\right)\cosh^2(a_k)\right) \tag{2.3.62}$$

This becomes independent of k and l precisely for $\eta=-2(N-2)$. So for this value of η we can regard K as a multiple of the identity. First let us consider all other values of η, so that K is equal to (2.3.36) with

$$A_k=\frac{1}{1-N+\sum_j\cosh^2(a_j)+\frac{1}{2}\left(2(N-2)+\eta\right)\cosh^2(a_k)}=\frac{1}{X+Y_k} \tag{2.3.63}$$

In order to satisfy the WDVV equations, the expression (2.3.37) should be symmetric in i and m. Just as in the previous section, the first nontrivial condition is that the fifth and sixth lines of (2.3.37) are exchanged under the interchange of i and m. This condition translates into

$$A_i\beta_{ji}\beta_{im}+A_j\beta_{ij}\beta_{jm}-A_m\beta_{jm}\beta_{im}=0 \tag{2.3.64}$$

and therefore

$$\begin{aligned}
(X+Y_j)(X+Y_m)\beta_{ji}\beta_{im}+(X+Y_i)(X+Y_m)\beta_{ij}\beta_{jm}-& \\
(X+Y_i)(X+Y_j)\beta_{jm}\beta_{im}=0&
\end{aligned} \tag{2.3.65}$$

Working this out further we find

$$-\frac{1}{16}\frac{(e^{4a_i}-1)(e^{4a_j}-1)(2(N-2)+\eta)^2}{e^{2(a_i+a_j)}}=0 \tag{2.3.66}$$

Therefore we find that for $\eta \neq -2(N-2)$ the WDVV equations are not satisfied. We will now determine what happens for the value $\eta = -2(N-2)$, for which K becomes a multiple of the identity. Then (2.3.37) becomes

$$
\begin{aligned}
\sum_{k=1}^{N} F_{ijk}F_{klm} \;\; = \;\; & \delta_{im}\Big(\beta_{ji}\beta_{lm} + \delta_{ij}\beta_{lm}M_i + \delta_{il}\beta_{ji}M_i + \delta_{ij}\delta_{lm}M_iM_m\Big) \\
+ \;\; & \delta_{jl}(1-\delta_{lm})\beta_{ij}\beta_{mj} \\
+ \;\; & \delta_{jl}\delta_{lm}M_l\beta_{im} \\
+ \;\; & \delta_{ij}\delta_{il}M_l\beta_{mi} \\
+ \;\; & \delta_{il}(1-\delta_{jm})\beta_{ji}\beta_{mi} \\
+ \;\; & \delta_{lm}(1-\delta_{ij})\left(\beta_{ji}\beta_{im} + \beta_{ij}\beta_{jm}\right) \\
+ \;\; & \delta_{jm}(1-\delta_{il})\beta_{ij}\beta_{lj} \\
+ \;\; & \delta_{ij}(1-\delta_{lm})\left(\beta_{li}\beta_{ml} + \beta_{mi}\beta_{lm}\right) \\
+ \;\; & \delta_{ij}\delta_{lm}\Big(M_i\beta_{im} + M_m\beta_{mi} + \sum_{k\neq i,m}\beta_{km}\beta_{ki}\Big) \\
+ \;\; & \delta_{jm}\delta_{il}\left(\beta_{im}^2 + \beta_{mi}^2\right)
\end{aligned}
\tag{2.3.67}
$$

The seventh and eighth lines are exchanged under the interchange of i and m for the same reasons as the fifth and sixth lines.Therefore it remains to check that the ninth and tenth lines are exchanged. To do this, we use (2.3.59) and (2.3.60) and find

$$
\begin{aligned}
& M_i\beta_{im} + M_m\beta_{mi} + \sum_{k\neq i,m}\beta_{km}\beta_{ki} = \\
& \quad \sum_{k\neq i,m}\left(\beta_{ik}\beta_{im} + \beta_{mk}\beta_{mi} + \beta_{km}\beta_{ki}\right) + \beta_{im}^2 + \beta_{mi}^2 + \eta(\beta_i\beta_{im} + \beta_m\beta_{mi}) = \\
& \qquad \sum_{k\neq i,m} 4 + \beta_{im}^2 + \beta_{mi}^2 + 2\eta = \beta_{im}^2 + \beta_{mi}^2 + 2\left(2(N-2)+\eta\right)
\end{aligned}
\tag{2.3.68}
$$

So for the special value $\eta = -2(N-2)$ we can conclude that F satisfies the generalized WDVV system. This finishes the proof of theorem 2.7. $\square$

2.3.1.4 An additional result

Here we mention a result which should be compared with theorem 2.4. It was obtained in the process of proving the main results of this section

Theorem 2.9. *The function (2.3.3) with $\alpha_- = 0$, $\alpha_+ = 1$, $\eta = 0$ and base function f_4 instead of f_5 satisfies the WDVV equations (1.2.2) if and only if*

$$Nb^3 + 3b^2c - ac^2 = 0 \tag{2.3.69}$$

Therefore adding Weyl invariant cubic terms

$$\frac{a}{6}\left(\sum_{i=1}^{N} a_i\right)^3 + \frac{b}{2}\left(\sum_{i=1}^{N} a_i\right)\left(\sum_{j=1}^{N} a_j^2\right) + \frac{c}{6}\sum_{i=1}^{N} a_i^3 \tag{2.3.70}$$

to the four-dimensional type A prepotential leads only to a solution of the generalized WDVV system if the condition (2.3.69) holds.

Proof. We will prove theorem 2.9 by adapting the proof of theorem 2.4. We find that $[F_i, F_m]$ is of the same form as (2.3.14) but with

$$\begin{aligned} \alpha &= -b^2 - ac - Nab \\ \beta &= Nb^2 + 2bc \end{aligned} \tag{2.3.71}$$

and we find precisely the same $K_{ij}K_{mn} - K_{mj}K_{in}$ as in (2.3.16). For $N \neq 2$ this again leads to a single relation, namely

$$Nb^3 + 3b^2c - ac^2 = 0 \tag{2.3.72}$$

which is to be compared with (2.3.20). This finishes the proof of theorem 2.9. □

2.3.2 Adding an extra variable

We take the perturbative prepotential (2.1.4) with base function (2.1.6) and add an additional variable a_0. We have the following result

Theorem 2.10. *[50] For any root system R on a space V with the standard Euclidean basis $\{e_1, ..., e_N\}$, the following function F*

$$F(a_0, ..., a_N) = \frac{1}{2}\sum_{\alpha\in R} f_5((\alpha, a)) + \gamma\left[\frac{1}{6}a_0^3 + \frac{1}{2}a_0(a, a)\right] \tag{2.3.73}$$

satisfies the WDVV system (1.2.2) for a particular value of γ which depends on the root system.

Since the γ turn out to be imaginary, the prepotentials no longer satisfy the property that substituting real variables a_k leads to a real value of F. To restore this property one can change the variables a_k to ia_k and the function f_5 to

$$f_5(x) = \frac{1}{6}(ix)^3 - \frac{1}{4}Li_3(e^{-2ix}) \tag{2.3.74}$$

This f_5 is also used in the literature, see e.g. [53].

Remark 2.11. *We can think of (a_0, a) as an element of an extension of the root space $\tilde{R} = \mathbf{R}e_0 \oplus R$ where we have introduced an additional basis vector e_0. With respect to the basis $\{e_0, ..., e_N\}$ we can define a flat metric on $\tilde{R}$ by means of the inner product $(e_i, e_j) = \delta_{ij}$, thus trivially extending the Euclidean metric on R. The matrix of third order derivatives F_0 naturally receives the interpretation as this metric. In fact, the variable a_0 plays the role of a special variable and the function $F(a_0, ..., a_N)$ satisfies the* original *WDVV system (1.1.2).*

Proof. We will use the matrix $K = F_0$ which obviously equals a multiple of the identity. Therefore the WDVV condition (1.2.2) reduces to

$$F_iF_m - F_mF_i = 0 \tag{2.3.75}$$

which are automatically satisfied whenever $i = 0$ or $m = 0$. Restricting ourselves to $i, m \neq 0$ the condition becomes

$$\sum_{k=1}^{N} (F_{ijk}F_{kmn} - F_{mjk}F_{kin}) + \gamma^2 (\delta_{ij}\delta_{mn} - \delta_{in}\delta_{jm}) = 0 \tag{2.3.76}$$

where all indices run from 1 to N. We remind the reader that the first of these two terms (but with base function f_4) also appears in the proof of the WDVV equations in the four-dimensional context, see (2.2.9). There we proved that this first term equals zero by using the Dunkl identity (2.2.12) corresponding to f_4. In the present five-dimensional situation we will use a similar Dunkl identity [51] associated with f_5:

$$\sum_{\substack{\sigma_\alpha\sigma_\beta = w \\ \alpha,\beta \in R_+}} B(\alpha,\beta) f_5'''(\alpha,a) f_5'''(\beta,a) = \sum_{\substack{\sigma_\alpha\sigma_\beta = w \\ \alpha,\beta \in R_+}} B(\alpha,\beta) \tag{2.3.77}$$

which is valid under the same conditions (2.2.13) and (2.2.14) as before. Since we use the same $B(\alpha,\beta)$ these conditions are again satisfied.

The underlying idea behind this second Dunkl identity is that $g(x) = f_4'''(x)$ satisfies the basic functional relation

$$\Big[g(x) + g(y)\Big] g(x+y) + \Big[g(x) - g(y)\Big] g(x-y) = 0 \tag{2.3.78}$$

whereas $g(x) = f_5'''(x)$ satisfies a similar relation

$$\Big[g(x) + g(y)\Big] g(x+y) + \Big[g(x) - g(y)\Big] g(x-y) = 2 \tag{2.3.79}$$

Just like we did in the four-dimensional situation we use the antisymmetry in j and n and restrict to positive roots. Taking the sum again over all roots the condition (2.3.76) becomes

$$\frac{1}{4}\sum_{w\in W} \sum_{\substack{\sigma_\alpha\sigma_\beta = w \\ \alpha,\beta \in R}} (\alpha,\beta) \left[\alpha_i\beta_m - \alpha_m\beta_i\right] \left[\alpha_j\beta_n - \alpha_n\beta_j\right] = -2\gamma^2 (\delta_{ij}\delta_{mn} - \delta_{in}\delta_{jm}) \tag{2.3.80}$$

We want to evaluate the left hand side. First we introduce the homogeneous 4-form A by

$$\frac{1}{4}A(x,y;u,v) = \sum_{\alpha,\beta\in R} (\alpha,\beta) \left[(\alpha,x)(\beta,y) - (\alpha,y)(\beta,x)\right] \left[(\alpha,u)(\beta,v) - (\alpha,v)(\beta,u)\right] \tag{2.3.81}$$

which is antisymmetric in x, y and in u, v. Moreover it is invariant under the Weyl group W and under permutation of x, y by u, v. Consequently a small calculation shows that we necessarily have

$$A(x,y;u,v) = c((x,u)(y,v) - (x,v)(y,u)) \tag{2.3.82}$$

	A_N	B_N	C_N	D_N	E_6	E_7	E_8	F_4
γ	$\sqrt{N+2}i$	$\sqrt{2(2N-3)}i$	$\sqrt{4(N+2)}i$	$\sqrt{4(N-2)}i$	$\sqrt{3}i$	$\sqrt{48}i$	$\sqrt{160}i$	$\sqrt{15}i$
c	$2(N+2)$	$4(2N-3)$	$8(N+2)$	$8(N-2)$	6	96	320	30

Table 2.1: *The numbers γ and c featuring in theorem 2.10 and its proof.*

for some fixed constant c. With respect to the Euclidean coordinates $e_1, \dots, e_n$ this means that the expression (2.3.80) equals

$$c\,(\delta_{ij}\delta_{mn} - \delta_{in}\delta_{jm})$$

Hence the WDVV condition reduces to $c = -2\gamma^2$. The precise values of c are evaluated with the help of appendix of Bourbaki [8] and they are listed in table 2.1. ☐

2.4 Energy scale and compactification radius as new variables

In this section we will see that there are natural parameters which can serve as extra variables for the four-dimensional as well as the five-dimensional prepotentials: in four dimensions it is the energy scale, in five dimensions a compactification radius. At first sight, there appear to be great advantages to this point of view: in four dimensions it has been suggested [6] that the energy scale provides an extra variable a_0 in such a way that $F(a_0, ..., a_N)$ satisfies the *original* WDVV system. This makes it very attractive to introduce an extra variable there. In five dimensions on the other hand we have seen in the previous section that we *need* an extra variable. To have a natural parameter play the role of this new variable, which we introduced by hand, is a very tempting idea. However, we will see that both in four as well as in five dimensions these ideas do not work.

We have already mentioned that in the four-dimensional situation, the perturbative prepotential (2.1.4) is the zero order term in the μ expansion of the full prepotential. More correctly, the prepotential doesn't have a power series expansion in μ, since it contains a $\log(\mu)$ term. This term however does not contribute to the WDVV equations since it is multiplied by a second order polynomial in the a variables. To be specific, the perturbative prepotential for a root system R including this $\log(\mu)$ term is given by

$$F(a_1, ..., a_N) = \frac{1}{2}\sum_{\alpha\in R}(\alpha, a)^2 \log\left(\frac{(\alpha, a)}{\mu}\right) \tag{2.4.1}$$

Giving degree 1 to both a and μ we find that F is homogeneous of degree 2. This can be expressed by means of Euler operators in the following way

$$\left[\mu\frac{\partial}{\partial\mu} + \sum_i a_i \frac{\partial}{\partial a_i}\right] F = 2F \tag{2.4.2}$$

The second order derivatives of F have degree zero, so we find that

$$\beta_{jk} := \mu\frac{\partial}{\partial\mu}F_{jk} = -\sum_i a_i F_{ijk} = -K_{jk} \tag{2.4.3}$$

where K_{jk} is the linear combination of third order derivatives of F that we used to prove the WDVV equations. The β_{jk} are physically very important objects called beta functions. For perturbative prepotentials they are the semiclassical coupling constants, whereas for the full prepotentials discussed in chapter 3 they determine the energy scale dependance of the coupling constants. Not only do we see that the β_{jk} naturally emerge in the proof of the WDVV equations, but moreover we find that by introducting a new variable $a_0 = -\log(\mu)$ we can write K_{jk} as

$$K_{jk} = F_{0jk} \tag{2.4.4}$$

which is similar to the five-dimensional situation. The question now arises whether or not the WDVV equations still hold if we regard a_0 as an extra variable on equal footing with the other a_i.

Proposition 2.12. *The function*

$$F(a_0, ..., a_N) = \frac{1}{2}\sum_{\alpha\in R}(\alpha, a)^2 \log((\alpha, a)) + a_0 \sum_{\alpha\in R}(\alpha, a)^2 \tag{2.4.5}$$

does not satisfy the WDVV equations (1.2.2) for any root system R.

Proof. Due to the dependence of F on a_0 none of the matrices F_k nor any of their linear combinations is invertible. ☐

Trying to repair this, one can try $a_0 = \mu$ as a new variable instead. The result however is equally disappointing.

Proposition 2.13. *The function*

$$F(a_0, ..., a_N) = \frac{1}{2}\sum_{\alpha\in R}(\alpha, a)^2 \log((\alpha, a)) - \log(a_0) \sum_{\alpha\in R}(\alpha, a)^2 \tag{2.4.6}$$

does not satisfy the WDVV equations (1.2.2) for any root system R.

Proof. Due to equation (2.4.2) we find that

$$(F_j)_{0k} = -\frac{1}{a_0}\sum_{i=1}^{N} a_i\, (F_j)_{ik} \tag{2.4.7}$$

so the zeroeth row of F_j can be expressed as a linear combination of the other rows.and therefore none of the F_j nor their linear combinations are invertible. ☐

We have found that adding μ as a new variable to the perturbative prepotential does not lead to new solutions to the WDVV equations. Since the full prepotential is also homogeneous of degree 2, the addition of μ doesn't give any solutions to the WDVV system there as well. This seems to contradict the findings in [6].

Let us now turn to the five-dimensional perturbative prepotentials. They are obtained by compactifying the fifth dimension of the theory and the dependence on the compactification radius R (not to be confused with the root system, also denoted by R) was suppressed so far. Including it[4], the prepotentials become [56]

$$F(a_1, ..., a_N) = \sum_{\alpha \in R} f_5((\alpha, a)) \tag{2.4.8}$$

with

$$f_5 = \frac{R}{6}x^3 - \frac{1}{4R^2}Li_3(e^{-2Rx}) \tag{2.4.9}$$

which reduces to (2.3.1) for $R = 1$. Regarding the compactification radius R as a new variable a_0 does not lead to the same F as in equation (2.3.73) in section 2.3.2. So even though it is tempting to believe that the extra variable that so naturally solves the problems in five-dimensional gauge theories is the compactification radius, this is in fact not true. Moreover, explicit computations on the type B prepotential have shown that (2.4.8) is not a solution to the WDVV equations.

4 Restoring the role of the compactification radius alters the analysis of section 2.3.2 only slightly: γ should be replaced by $R\gamma$.

Chapter 3

The full Seiberg-Witten prepotentials

Chapter 3

In this chapter we introduce the nonperturbative four-dimensional Seiberg-Witten prepotentials. Since their definition for a general simple Lie algebra is rather complex, we have chosen to first work out in section 3.1.1 the simplest example of Lie algebra A_N, which has the advantage of giving all the essential ingredients without going into numerous technicalities. The rest of section 3.1 deals with the main ingredients for the prepotentials for the other simple Lie algebras, which are subsequently defined in section 3.2.
Section 3.3 contains the proof of the important result that the nonperturbative prepotentials satisfy the generalized WDVV equations. The family of associative, commutative algebras (1.2.5) is identified and the relation (1.2.3) between its structure constants and the prepotential is shown to exist using two different methods.
Finally, in section 3.4 we show that in a certain limit the four-dimensional nonperturbative prepotential for type A_N Lie algebra goes to its perturbative counterpart. This establishes the link between the present and the previous chapter, as promised in section 2.1.

3.1 The Seiberg-Witten data

The full nonperturbative prepotentials originally arose as the solution to $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, also called Seiberg-Witten theory [59]. Although this physical context is essential for a full understanding of the prepotentials, it would take too much time to expose it here in full detail. For reviews on the subject, see for example [3, 7, 15].

On the other hand, the prepotentials can be described in the framework of an integrable system called the periodic Toda chain [22], [48]. We will assume that the reader has some knowledge of integrable dynamical systems, and we use the Toda chain context as the background and motivation for answering certain questions which are relevant in the construction of the prepotentials.

Apart from context, the prepotentials can be defined in purely mathematical terms with the help of the theories of Lie algebras and Riemann surfaces. Again we assume that the reader has a basic knowledge of both these theories. Restricting ourselves to this purely mathematical definition of the prepotential for a simple Lie algebra $\mathfrak{g}$, the three main ingredients in the construction are the following:

- The first ingredient is a family of Riemann surfaces corresponding to a set of affine curves

$$\Sigma_{\mathfrak{g}} = \left\{(x,z) \in \mathbf{C}^2 |\ P(x,z,u_1,...,u_N) = 0\right\} \tag{3.1.1}$$

  where the u_i serve as complex moduli parameters, N is the rank of $\mathfrak{g}$ and a Riemann surface in $\Sigma_{\mathfrak{g}}$ has genus $g \geq N$.

- The second ingredient is a special meromorphic differential λ_{SW} on $\Sigma_{\mathfrak{g}}$ which is called the Seiberg-Witten differential. Its special property is that the derivatives of λ_{SW} with respect to the moduli are holomorphic differentials on $\Sigma_{\mathfrak{g}}$.
- The third ingredient is a choice of $2N$ independent cycles on $\Sigma_{\mathfrak{g}}$ out of a total $2g$. If we choose a canonical basis $\{A_i, B_j\}$ of the first homology group, the choice consists of N cycles of type A and N cycles of type B in such a way that the restriction of the intersection form to this subset is nondegenerate: $A_i \circ B_j = \delta_{ij}$.

Once these ingredients are introduced, we define the prepotential in terms of period integrals of λ_{SW} over the chosen $2N$ cycles. Since varying the moduli will influence the period integrals, the (locally defined) prepotential is a function on moduli space.

3.1.1 A simple example: type A Lie algebra

Since the Seiberg-Witten data and the construction of the prepotential is complicated and technical for general simple Lie algebras, we give the example of Lie algebra $\mathfrak{g} = A_N$ here separately. As mentioned above, there are three main ingredients in the Seiberg-Witten data: a family of spectral curves, a special meromorphic differential on it and a set of $2N$ cycles.

3.1.1.1 The family of curves

A Riemann surface can be looked upon in various ways. Due to the Lie algebraic nature of our setup, we will often consider it as an algebraic curve in $\mathbf{P}^2$. On the other hand, we need the realization of the Riemann surface in terms of a complex manifold in order to study the holomorphic differentials on it. We will use the usual relation between these two realizations, see for example [10], [35].

Often we will give a Riemann surface in terms of an affine curve C in $\mathbf{C}^2$ defined through a polynomial P as

$$C = \left\{(x, y) \in \mathbf{C}^2 |\, P(x, y) = 0\right\} \tag{3.1.2}$$

The corresponding algebraic curve in $\mathbf{P}^2$ is given by adding the appropriate points at infinity. In terms of affine curves, a family of Riemann surfaces Σ is by definition

$$\Sigma = \left\{(x, y) \in \mathbf{C}^2 |\, P(x, y, u_1, ..., u_N) = 0\right\} \tag{3.1.3}$$

where for generic values of the complex parameters $u_1, ..., u_N$ the genus of the curve Σ is fixed to some number g. For special values however, the genus may decrease. Denoting by $\mathcal{M}$ the manifold $\mathbf{C}^N - \Delta$ with the special values of the u_i removed, we can look upon the family as a fibration of Riemann surfaces over $\mathcal{M}$. The space $\mathcal{M}$ is called the moduli space of the family and the u_i are called the moduli.

Returning to the specific example under consideration, the family of Riemann surfaces Σ_{A_N} is given by

$$\begin{aligned} \Sigma_{A_N} &= \left\{(x, y) \in \mathbf{C}^2 |\, P(x, y, u_i) = y^2 - W(x, u_i)^2 + 4 = 0\right\} && (3.1.4) \\ W(x, u_i) &= x^{N+1} + u_1 x^{N-1} + ... + u_{N-1} x + u_N && (3.1.5) \end{aligned}$$

Remark 3.1. *The fact that W can be identified with the Landau-Ginzburg superpotential of type A_N, or equivalently with a deformation of the type A_N singularity, is not restricted to the type A case. We will see in section 3.3.2 that the polynomial P defining the families of curves for the other ADE Lie algebras can also be used to construct a one-variable version of a Landau-Ginzburg superpotential. For the non ADE Lie algebras the curves are more complicated and there is no direct relation with the corresponding singularities.*

The curves in the family (3.1.4) are hyperelliptic, which makes their investigation relatively simple. Moreover, as a matter of fortunate coincidence in the type A_N case the rank N of the Lie algebra equals the genus g of the curves and these are the main reasons why it serves as the simplest example.

3.1.1.2 The moduli space and its Kähler metric

To get an idea of the structure of the moduli space $\mathcal{M}$, we mention that for all Lie algebras $\mathcal{M}$ is known to be a Kähler[1] manifold with Kähler metric defined in terms of the prepotential. If we denote the prepotential, which we will introduce later, by $\mathcal{F}(a_1, ..., a_N)$ then the metric is given in terms of the coordinates a_i by

$$(ds)^2 = \sum_{i,j} \text{Im}\left(\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}\right) da_i d\bar{a}_j \tag{3.1.6}$$

This relation is in fact the reason for the name prepotential, serving as the basic building block for the Kähler potential. In the context of perturbative prepotentials, we saw in section 2.2 that the linear combination K of third order derivatives of $\mathcal{F}$ appearing in the WDVV equations could be identified with a natural metric: the Killing form on the root space of the Lie algebra. We want to know if the Kähler metric can play a similar role, i.e. if there exist parameters α_k such that

$$\sum_k \alpha_k \mathcal{F}_{ijk} = \text{Im}\left(\mathcal{F}_{ij}\right) \tag{3.1.7}$$

Even though the parameters α_k are allowed to depend on $a_i, \bar{a}_i$, it still seems unlikely that they can link the holomorphic third order derivatives of $\mathcal{F}$ to the imaginary part of the second order derivatives. It is therefore very unlikely that the Kähler metric can fulfill a similar role as the Euclidean metric in the perturbative case.

3.1.1.3 The Seiberg-Witten differential and its derivatives

Moving on to the second ingredient in the construction of $\mathcal{F}$, the Seiberg-Witten differential λ_{SW} is given by

$$\lambda_{SW} = \log(y + W)dx - \log(2)dx \tag{3.1.8}$$

1 In fact, manifolds with Kähler metric of the form (3.1.6) are known as rigid special Kähler manifolds [13].

The special property of λ_{SW} is that its derivatives with respect to the moduli are all holomorphic. We will first explain what it means to differentiate (see [44]).

We can regard the equation $P(x,y,u) = 0$ as defining implicitly the function $y(x,u_k)$. The derivative of y with respect to the moduli gives

$$\frac{\partial y}{\partial u_k} = -\frac{P_{u_k}}{P_y} \tag{3.1.9}$$

where $P_{u_k} = \frac{\partial P}{\partial u_k}$. Using x as a local coordinate on the Riemann surface, we can extend this differentiation to differential forms $\omega = \phi dx$ by

$$\frac{\partial}{\partial u_i}(\phi dx) = \left(\frac{\partial \phi}{\partial u_i} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u_i}\right) dx \tag{3.1.10}$$

Alternatively, we can use y as a local coordinate and regard $P = 0$ as implicitly defining $x(y,u_i)$. We can calculate the derivative of $\omega = -\phi\frac{P_y}{P_x}dy$ again and see if we get the same answer as in (3.1.10). In general this is the case only up to total differential forms [44] so that taking a derivative of differential forms with respect to the moduli is unique only in cohomology.

Now we come back to the derivatives of λ_{SW}, which we will show to be cohomologous to a set of linearly independent holomorphic differentials. Using x as a local coordinate, the derivatives of λ_{SW} are

$$\frac{\partial \lambda_{SW}}{\partial u_k} = \frac{1}{y+W}\frac{\partial}{\partial u_k}(y+W)\,dx = \frac{1}{y+W}\left(\frac{W}{y}+1\right)\frac{\partial W}{\partial u_k}dx = x^{N-k}\frac{dx}{y} \tag{3.1.11}$$

and it is well-known that these give a basis of the holomorphic differentials of the hyperelliptic Riemann surfaces in the family Σ_{A_N}.

3.1.1.4 The special cycles

For a generic simple Lie algebra the rank is smaller than the genus of the family of curves and a selection of $2N$ of the $2g$ cycles has to be made. For type A_N no such selection is necessary, and therefore we can immediately proceed to define the prepotential.

3.1.1.5 The prepotential for type A Lie algebra

We define the period integrals of λ_{SW} over a set of canonical A cycles of the curve

$$a_i = \oint_{A_i} \lambda_{SW} \tag{3.1.12}$$

The a_i are moduli dependent and we can use their definition as a local change of variables on the moduli space. The Jacobian of this transformation is nonzero since

$$\frac{\partial a_i}{\partial u_j} = \oint_{A_i} \frac{\partial \lambda_{SW}}{\partial u_j} \tag{3.1.13}$$

and a matrix built from the integrals of all holomorphic differentials over all A cycles is always nondegenerate. Here we have pulled differentiation with respect to moduli through the integration sign. The justification for this is that the integral does not depend on the particular cycle A_i but only on its homology class. This allows to choose a representative of this class which encircles the branch cuts widely, so that changing the position of a branch point slightly doesn't change the cycle. This in turn allows to differentiate with respect to the moduli under the integration sign.

We can now define the derivatives of λ_{SW} with respect to the variables a_i by using the chain rule and we find that the $\frac{\partial \lambda_{SW}}{\partial a_i}$ form a canonical set of holomorphic differential forms since

$$\oint_{A_j} \frac{\partial \lambda_{SW}}{\partial a_i} = \frac{\partial a_j}{\partial a_i} = \delta_{ij} \tag{3.1.14}$$

We introduce the integrals of λ_{SW} over the B cycles

$$b_j = \oint_{B_j} \lambda_{SW} \tag{3.1.15}$$

Differentiating the b_j with respect to the moduli we find

$$\frac{\partial b_j}{\partial a_i} = \oint_{B_j} \frac{\partial \lambda_{SW}}{\partial a_i} = \Pi_{ij} \tag{3.1.16}$$

where Π_{ij} is the period matrix of the Riemann surface, which according to Riemann's bilinear relations is symmetric. Therefore we can (locally) integrate the b_j and obtain

$$b_j = \frac{\partial \mathcal{F}}{\partial a_j} \tag{3.1.17}$$

and the locally defined function $\mathcal{F}(a_1, ..., a_N)$ is called the prepotential.

Definition 3.2. *Associated to the type A_N Lie algebra, we define the family of curves Σ_{A_N} by (3.1.4) and a meromorphic differential λ_{SW} by (3.1.8). The prepotential $\mathcal{F}(a_1, ..., a_N)$ is defined locally on the moduli space $\mathcal{M}$ by*

$$\begin{aligned} a_i &= \oint_{A_i} \lambda_{SW} \\ b_j &= \oint_{B_j} \lambda_{SW} = \frac{\partial \mathcal{F}}{\partial a_j} \end{aligned} \tag{3.1.18}$$

Different choices of A and B cycles give different prepotentials, which we will put in one equivalence class for reasons described in the next subsection.

3.1.1.6 The effect of a particular choice of cycles

The fact that $\mathcal{F}$ cannot be extended to a global function on the moduli space was known already to Seiberg and Witten [59] for the simplest case of A_1. Instead of $\mathcal{F}$ being a function on $\mathcal{M}$, we have that (a_i, b_j) is a section of a flat bundle over $\mathcal{M}$ with structure group $\Gamma \subset \mathbf{C} \otimes Sp(2N, \mathbf{Z}) \times U(1)$.

Let us elaborate on this flat bundle. Since the moduli space $\mathcal{M}$ is constructed as a submanifold of $\mathbf{C}^N$, it will in general have a nontrivial fundamental group. One can circle along the nontrivial homotopy elements and pick up a monodromy on the cycles of the Riemann surface. Typically, the homology element encircles a gap of complex codimension one in $\mathbf{C}^N$ in which one or more cycles of the Riemann surface get pinched. The monodromy is given by the Picard-Lefschetz theorem, which prescribes that the effect of a pinched cycle δ on another cycle ζ is

$$\zeta \to \zeta + (\zeta \circ \delta)\delta$$

where $\circ$ denotes the intersection of the two. A small calculation shows that under these transformations a canonical homology basis remains canonical, in other words the monodromy operator is symplectic.

Together with the transformation on λ_{SW}, which may undergo a change in phase, this in turn induces a monodromy on the flat bundle sending (a, b) to some $(\tilde{a}, \tilde{b})$. The structure group of the bundle is therefore up to a phase generated by the monodromies, one for every nontrivial first homology element of $\mathcal{M}$. Since the monodromies are symplectic, the structure group is a subgroup of $\mathbf{C} \otimes Sp(2N, \mathbf{Z}) \times U(1)$.

The matrix of transformed variables

$$\frac{\partial \tilde{b}_j}{\partial \tilde{a}_i}$$

is again symmetric and can be integrated locally to a new function $\tilde{\mathcal{F}}(\tilde{a}_1, ..., \tilde{a}_N)$. This leads to different functions $\mathcal{F}$ locally for each patch of $\mathcal{M}$. In the physics literature, a lot of effort is spent on determining the precise cycles for each patch. Our point of view however concerns only the WDVV equations. Since we know from section 1.2.4 that both the symplectic group and the change of phase are symmetries of the WDVV equations, we are not so much interested in the particular local functions $\mathcal{F}$ since they will all be solutions to the WDVV equations. Therefore we will put all choices (and all resulting prepotentials) in one equivalence class.

3.1.2 Preliminaries: the periodic Toda chain and spectral curves

Consider a finite-dimensional dynamical system with enough preserved quantities in involution, so that there exists a canonical transformation to action-angle variables in which the time development of the system is given by a straight line motion on a torus. The existence of this so-called Liouville torus is guaranteed [42] if a Lax pair exists, i.e. a pair of square matrices L, M which depend on the positions and momenta in such a way that the equations of motion are equivalent to

$$\frac{dL}{dt} = [L, M] \tag{3.1.19}$$

In this case the preserved quantities are given by $\mathrm{Tr}(L^k)$, which are time independent due to the cyclicity properties of the trace. Here it is important that L and M are at least of dimension $N \times N$ if the phase space has dimension $2N$, so that there are enough functionally independent traces.

Sometimes, one can construct a Lax pair depending on an auxiliary parameter z: for any value of this so-called spectral parameter the pair $L(z), M(z)$ is a Lax pair for the system. The spectral equation for $L(z)$

$$P(x,z) = \det\Big[L(z) - x \cdot I\Big] = 0 \tag{3.1.20}$$

is then invariant under the flow. Equation (3.1.20) can be interpreted as the definition of a family of Riemann surfaces Σ, with the phase space variables playing the role of the moduli.

Since the Jacobian of the Riemann surface is a higher dimensional torus, it is tempting to suggest that the Liouville torus is the Jacobian of Σ. Indeed, for the periodic Toda chain which we will consider shortly this is sometimes the case. In the simplest case of a periodic Toda chain associated with the affine Lie algebra of type A_N, the Liouville torus is given precisely by the Jacobian of Σ for a particular Lax pair with spectral parameter [2, 1]. For other Lie algebras and other Lax pairs however, the genus of the spectral curves becomes too big and the Jacobian is larger than the Liouville torus. Still, one expects the Liouville torus to sit inside the Jacobian of Σ. The problem of finding the Liouville torus as a subvariety of the Jacobian of a spectral curve is called the Adler-van Moerbeke problem in the literature [2, 1], and its solution will be important in the definition of the prepotential and the proof of the WDVV equations. Essentially, the family of Riemann surfaces needed to define the prepotential will be given by (3.1.20) for the periodic Toda chain with a particular Lax pair, and the special subset of cycles together with λ_{SW} generate the Liouville torus inside the Jacobian.

Let us now turn to the periodic Toda chain, which was shown to be related to Seiberg-Witten theory in [22],[48]. The analysis of this section will follow closely that of [48]. The periodic Toda chain is a system that can be associated to any Lie algebra $\mathfrak{g}$. We will need a loop variable z in order to make contact with Seiberg-Witten theory, leading to the consideration of the affine Lie algebra $\mathfrak{g}^{(1)}$. In terms of the affine root system $R^{(1)}$ the Hamiltonian is given by

$$H = \frac{1}{2}\sum_{i=1}^{\text{rank }\mathfrak{g}} p_i^2 - \sum_{\alpha\in R^{(1)}} e^{-(\alpha,q)} \tag{3.1.21}$$

where $q = q_1\alpha_1 + ... + q_N\alpha_N$ is a linear combination of the simple roots. The dimension of the phase space therefore equals twice the rank of the Lie algebra. For any irreducible representation ρ of $\mathfrak{g}$ we can construct a Lax pair for the periodic Toda chain, and the matrix L which appears in the spectral curve is given by

$$\begin{aligned} L &= \rho(A) \\ A &= \sum_{i=1}^{\text{rank }\mathfrak{g}} (d_i h_i + c_i e_i + f_i) + z e_0 + \frac{c_0}{z} f_0 \end{aligned} \tag{3.1.22}$$

Here the e_i, f_i are the simple root generators of $\mathfrak{g}$ corresponding to α_i and $-\alpha_i$ respectively. The h_i are the elements of the Cartan subalgebra and e_0 is the highest root generator. The c_i, d_i are the so-called Flaschka coordinates on the phase space, obtained from the q_i, p_i in such a way that a certain product

$$\mu = \prod_{i=0}^{\text{rank }\mathfrak{g}} c_i^{n_i} \tag{3.1.23}$$

is time independent. This parameter μ will play the role of the energy scale, introduced in the previous chapter.

The dimension of any (faithful) irreducible representation ρ is bigger or equal to N, thus creating the possibility of existence of enough integrals of motion. Fixing the c_i for the moment, we find that the powers of traces of L are polynomials in the d_i. And what's more, these polynomials are invariant under the Lie algebra. The only functionally independent invariant polynomials in N variables are the N Casimir invariants, suggesting that the number of independent integrals of motion *precisely* equals half of the dimension of the phase space. One can check [48] that in general this is indeed the case.

After these preparations we are now ready to define the Seiberg-Witten data for the other simple Lie algebras, starting with the family of Riemann surfaces.

3.1.3 The Seiberg-Witten family of Riemann surfaces

Roughly speaking, the family of Riemann surfaces $\Sigma_{\mathfrak{g}}$ necessary for the Seiberg-Witten data is given by the spectral curve (3.1.20) for the periodic Toda chain, whose Hamiltonian is defined in terms of the affine Lie algebra $\mathfrak{g}^{(1)}$. Due to a physical requirement however, we should not consider the affine algebra $\mathfrak{g}^{(1)}$ but its dual $\left(\mathfrak{g}^{(1)}\right)^\vee$ which is obtained by exchanging long and short roots. For the simply laced algebras, the distinction is absent and we can continue directly. For the non-simply laced algebras, $\left(\mathfrak{g}^{(1)}\right)^\vee$ can be obtained from a simply laced algebra $\tilde{\mathfrak{g}}$ by dividing out an automorphism group π of $\tilde{\mathfrak{g}}$ [33]. In terms of the Dynkin diagram of $\tilde{\mathfrak{g}}$ the automorphism group consists either of reflections (A_{2N-1}, E_6, D_{N+1}) or rotations (D_4), see figure 3.1. The spectral curve (3.1.20) is now given in terms of the roots of $\tilde{\mathfrak{g}}$ which are invariant under π. For instance, instead of the highest (long) root of $\mathfrak{g}$, we now consider the highest (short) root of $\tilde{\mathfrak{g}}$ invariant under π.

Definition 3.3. *The family of Seiberg-Witten curves for four-dimensional $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge group $\mathfrak{g}$ is given by the spectral curve (3.1.20) associated with the periodic Toda chain for $\left(\mathfrak{g}^{(1)}\right)^\vee$ and the smallest representation ρ.*

Remark 3.4. *If Seiberg-Witten theory is to be related to the Toda system, the choice of representation (which does not appear in the definition of the Toda system itself) should be irrelevant. Indeed, we will find that not the spectral curve but the Liouville torus inside its Jacobian defines the prepotential. The choice of smallest representation is therefore just a matter of convenience.*

Before giving the curves explicitly for each simple Lie algebra, we can already see a lot of their structure. The Lax operator (3.1.22) can be assigned a natural degree by using the principal grading of the Lie algebra [33] and by assigning degrees $1, 2, h_{\mathfrak{g}}^\vee$ to d_i, c_i, z respectively, where $h_{\mathfrak{g}}^\vee$ is the dual Coxeter number of the Weyl group of $\mathfrak{g}$. This choice makes the Lax operator L homogeneous of degree 1. We denote this Lie algebraic degree of an object ϕ by $[\phi]_L$. The grading is respected by equation (3.1.20) and since this equation is Weyl invariant the coefficients of $x^k z^l$ in $P(x,z)$ are polynomials (of a particular degree) in the Casimir invariants u_k of $\mathfrak{g}$. Since there are $N = \text{rank}(\mathfrak{g})$ invariants, the spectral curve can be viewed as a family of curves depending on the N moduli u_k. Some Lie algebraic data is given in table 3.1.

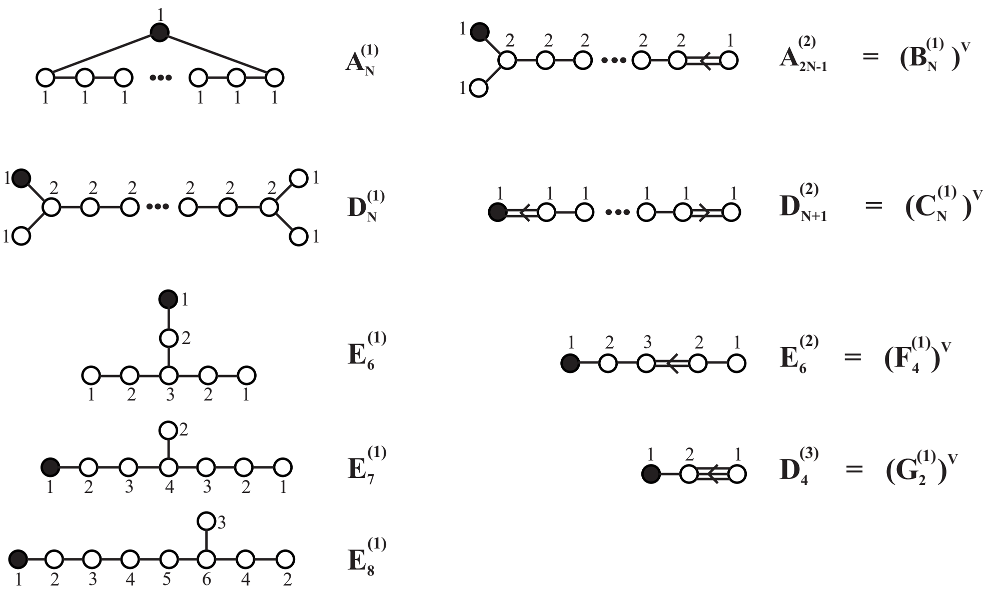

Figure 3.1: *The left side contains the affine Dynkin diagrams for simply laced Lie algebras, the right side shows the twisted affine Dynkin diagrams for non simply laced Lie algebras. These are obtained by dividing out the automorphism of the Dynkin diagram of the corresponding simply laced algebra. The affine roots are coloured black and the numbers n_i which occur in the definition (3.1.23) of μ are indicated for each root.*

Lie algebra $\mathfrak{g}$	$(\hat{g})^\vee$	$h_\mathfrak{g}$	$h_\mathfrak{g}^\vee$	exponents
A_N	$A_N^{(1)}$	$N+1$	$N+1$	$1, 2, ..., N$
B_N	$A_{2N-1}^{(2)}$	$2N$	$2N-1$	$1, 3, ..., 2N-1$
C_N	$D_{N+1}^{(2)}$	$2N$	$N+1$	$1, 3, ..., 2N-1$
D_N	$D_N^{(1)}$	$2N-2$	$2N-2$	$1, 3, ..., 2N-3, N-1$
E_6	$E_6^{(1)}$	12	12	$1, 4, 5, 7, 8, 11$
E_7	$E_7^{(1)}$	18	18	$1, 5, 7, 9, 11, 13, 17$
E_8	$E_8^{(1)}$	30	30	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	$E_6^{(2)}$	12	9	$1, 5, 7, 11$
G_2	$D_4^{(3)}$	6	4	$1, 5$

Table 3.1: *A list of the Coxeter numbers, dual Coxeter numbers and exponents of the simple Lie algebras. The degrees of the Casimirs are the exponents +1.*

It is convenient to view the spectral curve as a branched cover of the z sphere. For generic values of the moduli and z, the Lax operator $L = \rho(A)$ is the representation of a regular semisimple element A of the Lie algebra. This means that a Cartan subalgebra of $\mathfrak{g}$ can be defined by means of the centralizer of L. Since all Cartan subalgebras are conjugate, the element L is conjugate to an element $v(z) \cdot h = \sum_{i=1}^{N} v_i(z) h_i$ in the standard Cartan subalgebra. The eigenvalues x of $\rho(L(z))$ are therefore given by $x = v(z) \cdot \omega_k$ where the ω_k denote the weights of the representation. The spectral curve can now be denoted by

$$P(x,z) = \prod_{k=1}^{\dim \rho} (x - v(z) \cdot \omega_k) = 0 \tag{3.1.24}$$

If the dimension of the weight space of one of the weights ω is more than one-dimensional, we remove all but one factor $x - v(z) \cdot \omega$. Since the weights form a Weyl invariant subset of the root space, the spectral curve splits according to their Weyl orbits. Representations with only one Weyl orbit of weights are called miniscule. If the representation is not miniscule, we focus on the piece containing the highest weight. While discussing the general scenario, we will assume that this piece is nonsingular[2].

We will now discuss the pieces of plumbing that connect the different sheets of the foliation, starting with the finite values of x. For generic values of z, we know that $L(z)$ is a regular semisimple element of $\mathfrak{g}$ conjugate to $v(z) \cdot h$. By using the action of the Weyl group, we can take $\mathrm{Im}(v) \cdot h$ to be in the fundamental Weyl chamber. Branch points of the curve occur for those z for which $\frac{\partial P}{\partial x} = 0$, in other words if two eigenvalues of $\rho(L)$ come together. This happens for example when $v(z) \cdot h$ hits a wall of the fundamental Weyl chamber, i.e. when $v(z) \cdot \alpha_k = 0$ for some simple root α_k. If this is the case, then the weight ω_i and its reflection $\omega_j = \sigma_{\alpha_k} \omega_i$ give the same eigenvalue since

$$v \cdot \omega_j = v \cdot \sigma_{\alpha_k} \omega_i = \sigma_{\alpha_k} v \cdot \omega_i = v \cdot \omega_i \tag{3.1.25}$$

From the expression (3.1.22) for the Lax operator one finds [48] that the curves exhibit a symmetry $z \to \frac{\mu}{z}$ where μ was defined in (3.1.23), see also figure 3.1. Therefore the branch points come in pairs to form square root branch cuts. There can also be other branch points or even singular points for which $v(z) \cdot h$ does not hit a wall of the fundamental Weyl chamber, and these points are called accidental. We will assume that there are none of these points (it can be checked explicitly in each case) but even if there are, it is possible to create a cover of the curve in such a way that the accidental branch points and singularities are removed. This cover is discussed in detail in section 3.5.

The preceding recipe tells us how to connect the sheets of the cover for finite values of z. For $z = 0$ and $z = \infty$ there is also a good description of what happens in terms of the root system of $\mathfrak{g}$. On the $\mathbf{P}^1$ base on which z takes its values we have given branch points $z_i^{\pm}$ corresponding to each simple root α_i of $\mathfrak{g}$, whose various lifts to the sheets of the foliation make up the branch cuts for finite values of x. Of course any lift of a closed curve C on the z sphere encircling all the branch points must come back to the sheet it started on since we can deform C to a trivial curve on the z sphere. Due to the symmetry $z \to \frac{\mu}{z}$ any lift of the closed curve C' in figure 3.2 must also come back to the same sheet. Adding C and C' we see that any lift of a closed curve encircling all the z_i^- and $z = 0$ must also come back to the same sheet, so that encircling only $z = 0$ has the same effect as encircling all the branch points z_i^-.

2 Actually, this is not a reasonable assumption. Usually the curves are singular but there exists a natural desingularisation which one should study instead, see section 3.5

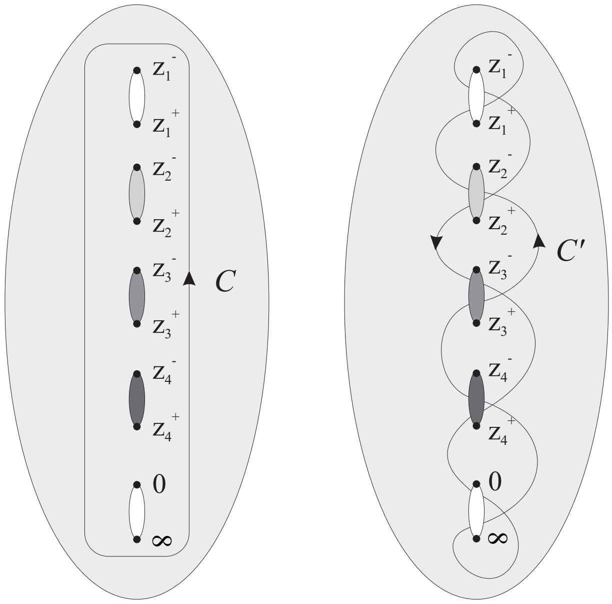

Figure 3.2: *The z-sphere is given twice for A_4 together with the branch points: $z = 0, \infty$ and the $z_i^{\pm}$. The curve C in the left picture is trivial and is therefore closed when lifted to the Riemann surface. Since all branch cuts are hyperelliptic the same is true for C' in the right picture.*

Therefore, starting on the sheet S_ω with weight ω and then making a circle around $z = 0$, one ends up on the sheet with weight $s\omega$ where s is the Coxeter element of the Weyl group of $\mathfrak{g}$. So the branch cut between $z = 0$ and $z = \infty$ connects all the sheets whose weights are in one orbit of the cyclic group $\mathbf{Z}_{h_g^\vee}$ generated by s.

In figure 3.3 we have given the example of Lie algebra A_4 in the 10-dimensional representation [48]. The weights are given for each sheet, and two sheets are connected above the α_i cut if and only if their weights are exchanged under σ_{α_i}. The Coxeter element s splits the weights into two groups of 5, which specifies how the sheets are connected at infinity. The genus of the curve is thus $g = 11$, which is the same answer as one gets from a direct calculation using the equation for the spectral curve given in (3.3.63). This shows that there are no accidental points.

As another example, we consider again A_4 but now in the 24-dimensional adjoint representation. This representation is not miniscule, because the weights split into two disjoint Weyl orbits: the roots of A_4 each of which has multiplicity 1, and the zero vector which has multiplicity 4. Consequently the Riemann surface splits into two parts, and we concentrate on the part containing the highest weight and having degree 20. The genus of the curve is now $g = 25$, see figure 3.4. Again accidental points are absent since a direct calculation of the genus using the spectral curve (3.3.63) gives the same result.

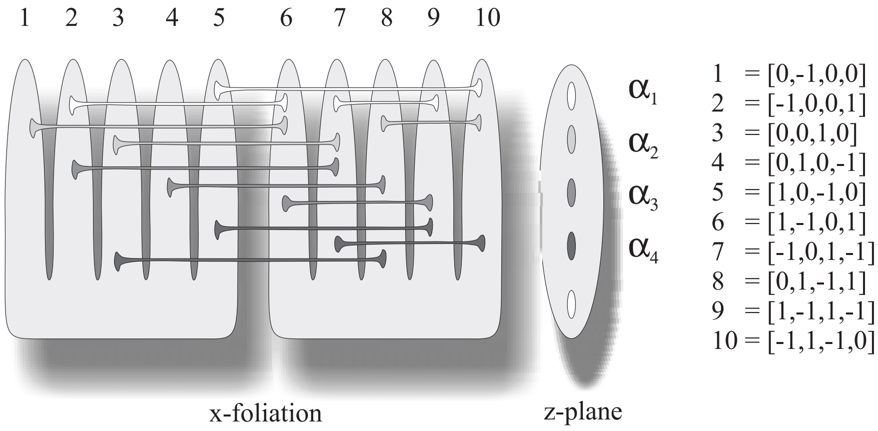

Figure 3.3: *The Riemann surface for A_4 in the antisymmetric 10-dimensional representation. The genus of the curve is $g = 11$ and we have labeled the weights by their coefficients in terms of the fundamental weights. Picture taken from [48]*

The list of Seiberg-Witten curves is [48],[30]

$$\begin{aligned}
&A_N && z+\frac{\mu}{z}+x^{N+1}+u_1x^{N-1}+...+u_N=0 \\
&B_N && x(z+\frac{\mu}{z})+x^{2N}+u_1x^{2N-2}+u_2x^{2N-4}...+u_N=0 \\
&C_N && \left(z-\frac{\mu}{z}\right)^2+x^2\left(x^{2N}+u_1x^{2N-2}+u_2x^{2N-4}...+u_N\right)=0 \\
&D_N && x^2\left(z+\frac{\mu}{z}\right)+x^{2N}+u_1x^{2N-2}+...+u_{N-2}x^4+u_Nx^2+u_{N-1}^2=0 \\
&E_6 && \frac{1}{2}x^3(z+\frac{\mu}{z}+u_6)^2-q_1(x)(z+\frac{\mu}{z}+u_6)+q_2(x)=0 \\
&F_4 && -8\left(z+\frac{\mu^2}{z}\right)^3+s_1(x)\left(z+\frac{\mu^2}{z}\right)^2+s_2(x)\left(z+\frac{\mu^2}{z}\right)+s_3(x)=0 \\
&G_2 && 3\left(z-\frac{\mu}{z}\right)^2-x^8+2ux^6-\left[u^2+z+\frac{\mu}{z}\right]x^4+\left[v+2u\left(z+\frac{\mu}{z}\right)\right]x^2=0
\end{aligned} \tag{3.1.26}$$

Although the prepotential for G_2 depends only on two variables and therefore trivially satisfies the WDVV equations, we have included the Seiberg-Witten curves for G_2 in the list. The curves for E_7 and E_8 have been omitted because they are big and cumbersome. The expressions for $s_i(x), q_i(x)$ can be found in appendix A. Note how for simply laced Lie algebras the z dependence is characterized by

$$P(z+\frac{\mu}{z},x,u_1,...,u_N)=P(z+\frac{\mu}{z}=0,x,u_1,...,u_N+z+\frac{\mu}{z}) \tag{3.1.27}$$

As we will see, there is a direct relation between the A-D-E Seiberg-Witten curves for any representation on the one hand and the A-D-E Landau-Ginzburg superpotentials [60, 16] or miniversal deformations of isolated singularities [5] on the other hand. The equation (3.1.27) helps establish this relation, and the twisting procedure necessary to define the Seiberg-Witten curves for the non simply laced Lie algebras disturbes it. If it wasn't for this twisting, there would be a relation with the corresponding singularities [64].

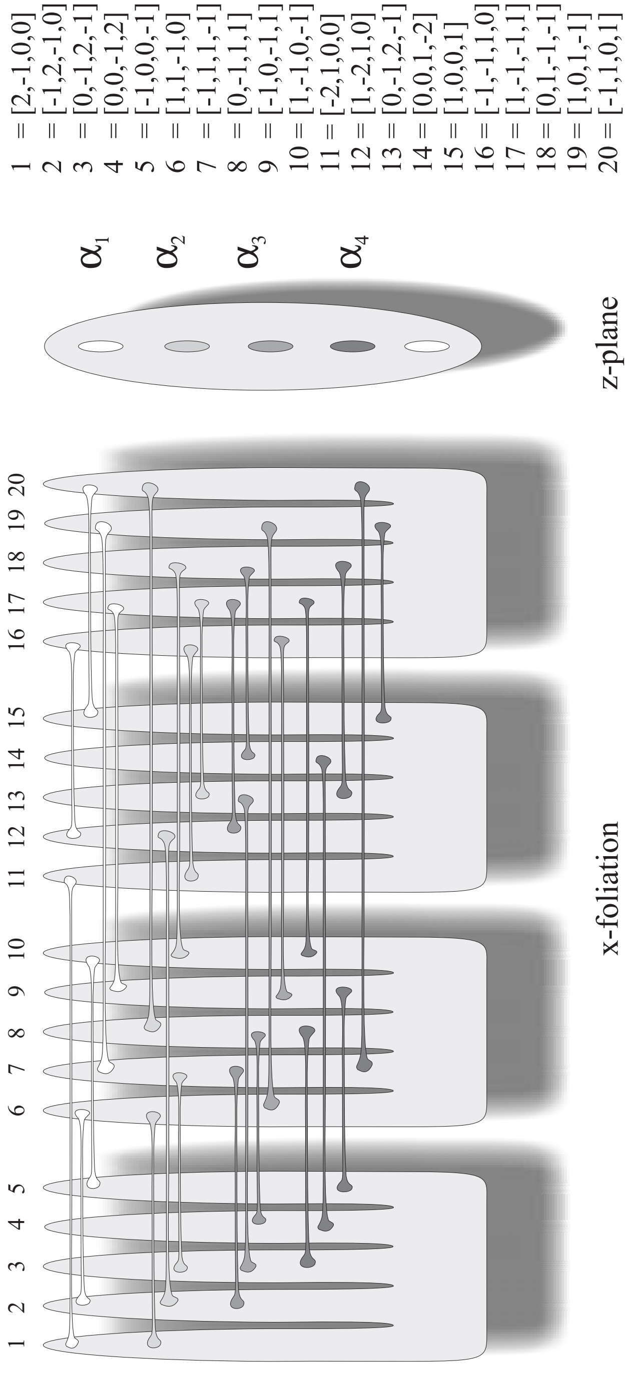

Figure 3.4: *The Riemann surface for A_4 in the 24-dimensional adjoint representation. Since the spectral curve splits into two parts, we have concentrated on the part containing the highest weight. The genus of the curve is $g = 25$. As usual we have labeled the weights by their coefficients with respect to the fundamental weights.*

$\mathfrak{g}$	A_N	B_N	C_N	D_N	E_6	F_4	G_2
g	N	$2N-1$	$2N$	$2N-1$	34	46	11

Table 3.2: *The genera for the Seiberg-Witten curves of ADE type*

For the classical Lie algebras there exists a change of variables that give the curves in the following standard hyperelliptic form (see also section 3.1.1)

$$\begin{aligned}
\text{Type } A_N: \quad y &= z - \frac{\mu}{z} \\
y^2 &= \left(x^{N+1} + u_1 x^{N-1} + u_2 x^{N-2} + ... + u_N\right)^2 - 4\mu \\
\text{Type } B_N: \quad y &= x\left(z - \frac{\mu}{z}\right) \\
y^2 &= \left(x^{2N} + u_1 x^{2N-2} + u_2 x^{2N-4} ... + u_N\right)^2 - 4\mu x^2 \\
\text{Type } C_N: \quad y &= \frac{1}{x}\left(z^2 - \frac{\mu^2}{z^2}\right) \\
y^2 &= \left(x^{2N} + u_1 x^{2N-2} ... + u_N\right)\left(x^2\left(x^{2N} + u_1 x^{2N-2} ... + u_N\right) + 4\mu\right) \\
\text{Type } D_N: \quad y &= x^2\left(z - \frac{\mu}{z}\right) \\
y^2 &= \left(x^{2N} + u_1 x^{2N-2} + ... + u_{N-2} x^4 + u_N x^2 + u_{N-1}^2\right)^2 - 4\mu x^4
\end{aligned}$$

The curves for E_6, F_4 and G_2 however are *not* hyperelliptic. For generic values of the moduli u_i all curves within one family have the same genus, and a list of these genera is given in table 3.2.

As a final example, consider figure 3.5 where the curve for E_6 is depicted. The 27 weights are labeled by the coefficients in the expansion in terms of fundamental weights, so $[1, 0, 0, 0, 0, 0]$ stands for the fundamental dominant weight λ_1 which is also the highest weight for this representation. Each weight has multiplicity one, the 27 sheets are connected at $z = 0, \infty$ by the Coxeter element and the orbits have dimension 12, 12 and 3. Above each simple root there are 6 square root branch cuts, giving the Riemann surface genus 34 which is the same as the value found in table 3.2. This shows that there are no accidental points.

3.1.4 The Seiberg-Witten differential and its derivatives

The second ingredient of the Seiberg-Witten data is a special meromorphic differential λ_{SW}.

Definition 3.5. *The Seiberg-Witten differential λ_{SW} is given by*

$$\lambda_{SW} = \log(z)dx = d\left(x\log(z)\right) - x\frac{dz}{z} \simeq -x\frac{dz}{z} \tag{3.1.28}$$

where $\simeq$ denotes equality modulo total differentials.

Since we will mainly be interested in the period integrals of λ_{SW}, only its cohomology class is important. In the specific case of Lie algebra A_N, the differential form (3.1.8) reduces to

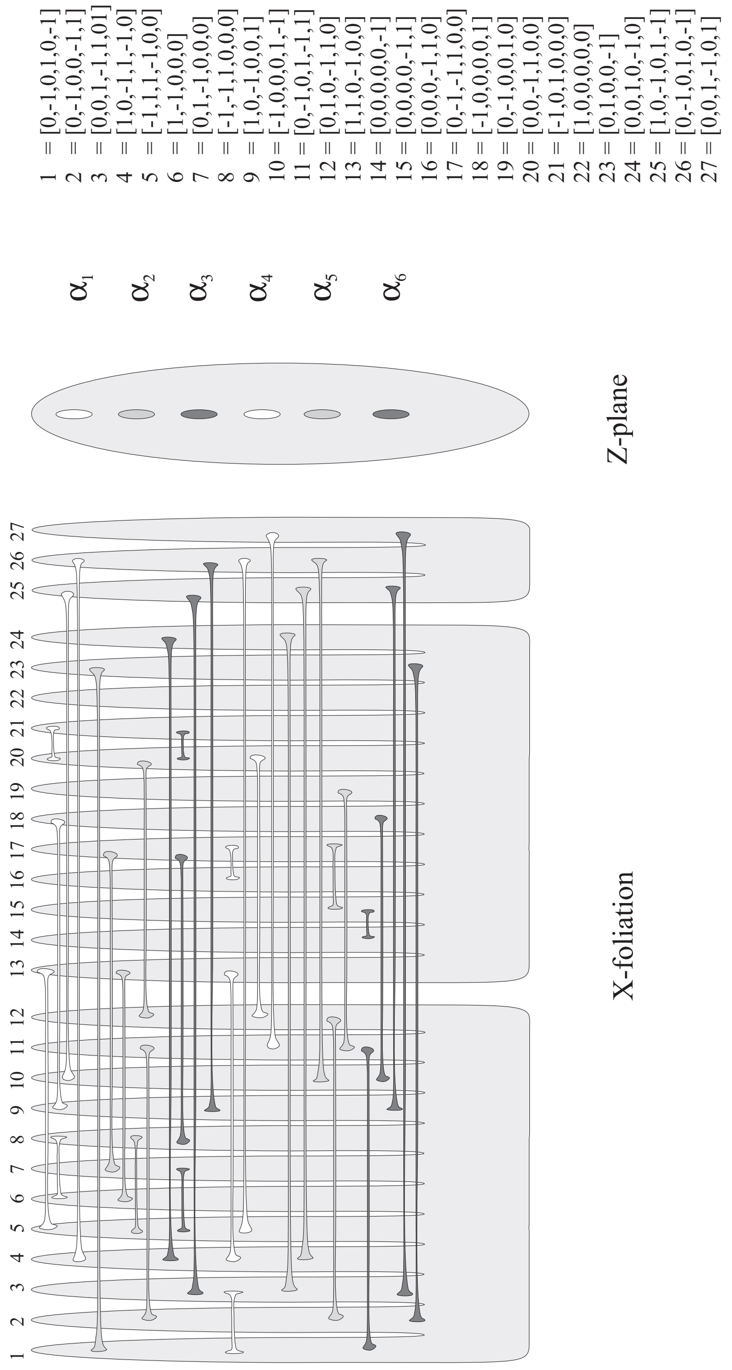

Figure 3.5: *The Riemann surface for E_6 in the 27-dimensional representation. In the z-plane the branch cuts are depicted according to the six simple roots of E_6 (in standard notation) and the cut from $z = 0$ to ∞ is omitted. Above each root there are six pieces of plumbing connecting the three Coxeter orbits. The genus of the curve is $g = 34$.*

(3.1.28) since

$$\log(y+W)dx - \log(2)dx = \log(z - \frac{\mu}{z} + z + \frac{\mu}{z})dx - \log(2)dx = \log(z)dx \quad (3.1.29)$$

In terms of the Toda system, λ_{SW} plays the role of the action differential pdq [48]. The main special property of λ_{SW} that we are interested in is that its derivatives with respect to the moduli parameters u_k give holomorphic differentials.

3.1.4.1 Holomorphic differentials

In this section, we describe the construction of a basis of holomorphic differential forms on any Riemann surface, see e.g. [10], [35]. Let the Riemann surface be given by an affine equation

$$P(x,z) = 0 \quad (3.1.30)$$

In particular, we are interested in the affine curves obtained from the Seiberg-Witten family (3.1.26). In order to make those curves affine, we multiply them with a monomial z^k of minimal degree to make P a polynomial. Viewing the curve as defining implicitly $x(z)$, the branch points are given by $P_x = 0$ and $P_z \neq 0$. Consider the differential form

$$\omega = \frac{\phi(x,z)dz}{P_x} = -\frac{\phi(x,z)dx}{P_z} \quad (3.1.31)$$

Denoting the degree[3] of P by $[P] = d$, one finds that for ϕ a polynomial of degree smaller or equal to $d-3$, the differential form ω is nonsingular for all points except the singularities. In particular, ω is nonsingular in the branch points and due to the condition on the degree of ϕ also at infinity. If there are no singular points, a basis of holomorphic forms can be constructed from the ω as above, and their number is $\frac{1}{2}(d-1)(d-2)$ which is in accordance with the degree-genus formula for nonsingular curves (see e.g. [35]).

We will first check that the derivatives of λ_{SW} are holomorphic outside the singular points. Denote the Seiberg-Witten curves by

$$P(x,z,u) = \sum_{i=0}^{r} \left(z^2+\mu\right)^i z^{r-i} q_i(x,u) \quad (3.1.32)$$

and the degree of P is given by

$$[P] = [q_0] + r \quad (3.1.33)$$

The derivatives of λ_{SW} with respect to the moduli are given by

$$\frac{\partial \lambda_{SW}}{\partial u_k} = -\left(\frac{\partial}{\partial u_k}x\right)\frac{dz}{z} = \frac{P_{u_k}}{z}\frac{dz}{P_x} \quad (3.1.34)$$

It can be checked explicitly for every Seiberg-Witten curve in (3.1.26) that q_r is moduli independent. Hence $\frac{P_{u_k}}{z}$ is a polynomial and taking into account that it is homogeneous in

3 Note the difference between the notation $[.]$ of degrees of polynomials in terms of their variables and the Lie algebraic degree $[.]_L$.

terms of the Lie algebraic grading, in which z has $h_{\mathfrak{g}}^{\vee}$ times the degree of x, we find that its polynomial degree is

$$\left[\frac{P_{u_k}}{z}\right] \leq [q_0] + r - 1 - [u_k]_L = d - 1 - [u_k]_L \tag{3.1.35}$$

and since the u_k are the Casimir invariants of the Lie algebra, their Lie algebraic degree is bigger or equal to 2. Therefore the derivatives of λ_{SW} are holomorphic for nonsingular curves.

The restrictions that follow from the singularities are straightforward. In the affine coordinate patch (not at infinity) one can write $x(z)$ as a convergent power series if $P_x \neq 0$ using the implicit function theorem. For singular points, using the method of Puiseux expansions one can write $x(z)$ as a *fractional* power series instead, with a number of different series for each individual singularity [35]. The form ω should be nonsingular when each of these fractional power series is substituted into it. The singular points which are at infinity are treated in the same way after a change of variables in $\mathbf{P}^2$ to the relevant coordinate patch.

For the classical Lie algebras we have given the curves in standard hyperelliptic form in (3.1.28) from which it is easy to see that the derivatives of λ_{SW} are holomorphic. For E_6, F_4 and G_2 explicit computations were done using the computer algebra package Maple, which show that the derivatives of λ_{SW} are nonsingular not only in the branch points of the curve and at infinity but even in its singular points. We therefore arrive at the following proposition

Proposition 3.6. *The derivatives of λ_{SW} with respect to the moduli are holomorphic for all simple Lie algebras.*

As an example, we consider the curve of G_2 of genus 11, given in (3.1.26). The 11 holomorphic forms are given by $\frac{\phi_k(x,z)dz}{P_x}$ and a list of the ϕ_k is given below:

$$\{\phi_k\} = \left\{x^6 z, x^5 z, x^4 z, x^3 z, x^2 z^2, x^2 z, x^2, xz, x, xz^2, 1 - z^2\right\} \tag{3.1.36}$$

On the other hand, the derivatives of λ_{SW} are given by

$$\begin{aligned}
\frac{\partial \lambda_{SW}}{\partial u} &= \frac{P_u dz}{z P_x} = \left(2x^6 z - 2ux^4 z + 2x^2 z^2 + 2x^2\right) \frac{dz}{P_x} \\
\frac{\partial \lambda_{SW}}{\partial v} &= \frac{P_v dz}{z P_x} = x^2 z \frac{dz}{P_x}
\end{aligned} \tag{3.1.37}$$

and can be written as linear combinations of the holomorphic forms.

3.1.5 The subset of cycles

The third and final ingredient of the Seiberg-Witten data is a special subset of $2N$ independent cycles. For A_N in the fundamental representation one can take all cycles and no selection is necessary. The Seiberg-Witten curves of the other classical Lie algebras in the fundamental representation possess an involution which makes it easy to identify the special cycles. For the remaining cases there exists a more general method [48],[29] based on the action of the Weyl group on the curves. Here we treat only the simply laced Lie algebras, referring the reader to [29] for the non simply laced ones.

3.1.5.1 The special cycles for the B, C, D Lie algebras

We regard the curves in their hyperelliptic form (3.1.28). Each of them has the involution $\sigma(x) = -x$. This helps us to identify the special cycles immediately: consider the curves as defining implicitly $y(x)$, and draw the branch cuts in the x-plane in such a way that the cuts come in pairs $K_i^{\pm}$ related by σ. We denote the counterclockwise contour around $K_i^{\pm}$ on the first sheet by $C_i^{\pm}$. The special A cycles are then defined by

$$A_i = C_i^+ - C_i^- \tag{3.1.38}$$

The special B cycles are the obvious ones going from K_i^- to K_i^+ on the first sheet and back again on the second, without intersecting each other.

3.1.5.2 Cycles for simply laced Lie algebras

Here we will discuss the more general method of identifying the special cycles, based on the action of the Weyl group on the family of curves as discussed in section 3.1.3. This method is independent of the particular representation used to define the Seiberg-Witten curves and it solves the Adler-van Moerbeke problem of identifying the Liouville torus inside the Jacobian of the Toda spectral curve for any representation.

First we note that any lift A_i^ω of a counterclockwise closed contour C_i around only the α_i cut on the z sphere to the sheet S_ω labeled by the weight ω is a closed curve on that sheet. If $\alpha_i \cdot \omega = 0$ then A_i^ω is trivial, otherwise it's not. Since the branch cuts come in pairs, the cycle A_i^ω is homologous to $-A_i^{\sigma_{\alpha_i}\omega}$. By multiplying the contribution of each cycle by $\omega \cdot \alpha_i$ the contributions from the two different sheets add up since $\sigma_{\alpha_i}\omega \cdot \alpha_i = -\omega \cdot \alpha_i$. It is convenient to introduce the combinations

$$\hat{A}_i^\omega = \frac{1}{2}\left(A_i^\omega - A_i^{\sigma_{\alpha_i}\omega}\right) \tag{3.1.39}$$

These are the building blocks of the A cycles.

Definition 3.7. *The special A cycles are given by*

$$A_i = N_{i,\rho}\sum_\omega (\omega \cdot \alpha_i) A_i^\omega = N_{i,\rho}\sum_\omega (\omega \cdot \alpha_i)\hat{A}_i^\omega \tag{3.1.40}$$

where A_i^ω is the lift of C_i to the sheet characterized by the weight ω. The absolute value of $\omega \cdot \alpha_i$ determines how many times to wind around the cut and its sign determines in what direction to wind: a positive value means anti-clockwise and negative means clockwise. The normalisation factor $N_{i,\rho}$ is given by

$$N_{i,\rho} = \frac{1}{\sum_\omega |(\omega \cdot \alpha_i)|^2} \tag{3.1.41}$$

On the other hand, we need a set of B cycles. To define the cycle B_i, we draw a number of lifts B_i^ω to the sheet S_ω of the open curve D_i going from $z = 0$ to $z = z_i^-$ on the z sphere. The number and direction of the lifts is again determined by $\omega \cdot \alpha_i$: for example, $\omega \cdot \alpha_i = 1$ means one strand going up from $z = 0$ to $z = z_i^-$ on S_ω, while $\omega \cdot \alpha_i = -2$ means two strands going down from $z = z_i^-$ to $z = 0$ (see figure 3.6). Then for each Coxeter orbit $\mathcal{O}_k$ of sheets, we connect the strands through the cuts between $z = 0$ and $z = \infty$. To prove that this gives a closed curve B_i, we note that the number of strands going down to $z = 0$ on $\mathcal{O}_k$ equals the number of strands going up, since $\sum_{\omega \in \mathcal{O}_k} (\omega \cdot \alpha_i) = (\sum_{\omega \in \mathcal{O}_k} \omega) \cdot \alpha_i = 0$. Therefore we can connect the strands on every Coxeter orbit, which shows that B_i is indeed closed. Again, it is convenient to introduce the linear combination

$$\hat{B}_i^\omega = \frac{1}{2}\left(B_i^\omega - B_i^{\sigma_{\alpha_i}\omega}\right) \tag{3.1.42}$$

We are now ready to define the special B cycles, see also figure 3.6.

Definition 3.8. *The special B cycles are given by*

$$B_i = N_{i,\rho} \sum_\omega \omega \cdot \alpha_i B_i^\omega = N_{i,\rho} \sum_\omega \omega \cdot \alpha_i \hat{B}_i^\omega \tag{3.1.43}$$

where B_i^ω is the lift of the open curve D_i to the sheet S_ω. The number $\omega \cdot \alpha_i$ decides on the direction and number of strands. The curve is then closed up through the cuts between $z = 0$ and $z = \infty$.

The normalisation factor $N_{i,\rho}$ is chosen in such a way that the period integrals of λ_{SW} are representation independent: on S_ω we have $\lambda_{SW} = -v(z) \cdot \omega \frac{dz}{z}$ due to (3.1.28) and therefore

$$\begin{aligned}
\oint_{A_i} \lambda_{SW} &= N_{i,\rho} \sum_\omega \omega \cdot \alpha_i \oint_{\hat{A}_i^\omega} \lambda_{SW} = N_{i,\rho} \sum_{\omega \cdot \alpha_i > 0} \omega \cdot \alpha_i \oint_{(A_i^\omega - A_i^{\sigma_{\alpha_i}\omega})} \lambda_{SW} \\
&= N_{i,\rho} \sum_{\omega \cdot \alpha_i > 0} \omega \cdot \alpha_i \oint_{C_i} (-v(z) \cdot \omega + v(z) \cdot \sigma_{\alpha_i}\omega) \frac{dz}{z} \\
&= -N_{i,\rho} \sum_{\omega \cdot \alpha_i > 0} \frac{(\omega \cdot \alpha_i)^2}{\alpha_i \cdot \alpha_i} \oint_{C_i} 2v(z) \cdot \alpha_i \frac{dz}{z} \\
&= -N_{i,\rho} \sum_\omega \frac{(\omega \cdot \alpha_i)^2}{\alpha_i \cdot \alpha_i} \oint_{C_i} v(z) \cdot \alpha_i \frac{dz}{z} \\
&= \frac{-1}{\alpha_i \cdot \alpha_i} \oint_{C_i} v(z) \cdot \alpha_i \frac{dz}{z}
\end{aligned} \tag{3.1.44}$$

which is indeed representation independent. A similar reasoning shows that the period integrals of λ_{SW} over the B cycles are independent of ρ. This is also true for the non simply laced Lie algebras [29].

To show that the A and B cycles just defined have the proper intersection numbers, we proceed as follows. It is clear that $A_i \circ A_j = B_i \circ B_j = 0$ and $A_i \circ B_j = -B_j \circ A_i = \gamma_i \delta_{ij}$ for some number γ_i. To determine the γ_i, we count the intersection on each sheet S_ω. Up to the normalisation, the number of strands from the B cycle that cross the closed curve from the A cycle is $|\omega \cdot \alpha_i|$ and there are also $|\omega \cdot \alpha_i|$ copies of the A cycles. Since the contribution to the intersection is always positive we find that the contribution from the sheet S_ω is $|(\omega \cdot \alpha_i)|^2$. Summing the contributions for all sheets and taking into account the normalisation we find

$$A_i \circ B_j = (N_{i,\rho})^2 \sum_\omega |\omega \cdot \alpha_i|^2 \delta_{ij} = \frac{1}{\sum_\omega (\omega \cdot \alpha_i)^2} \delta_{ij} \tag{3.1.45}$$

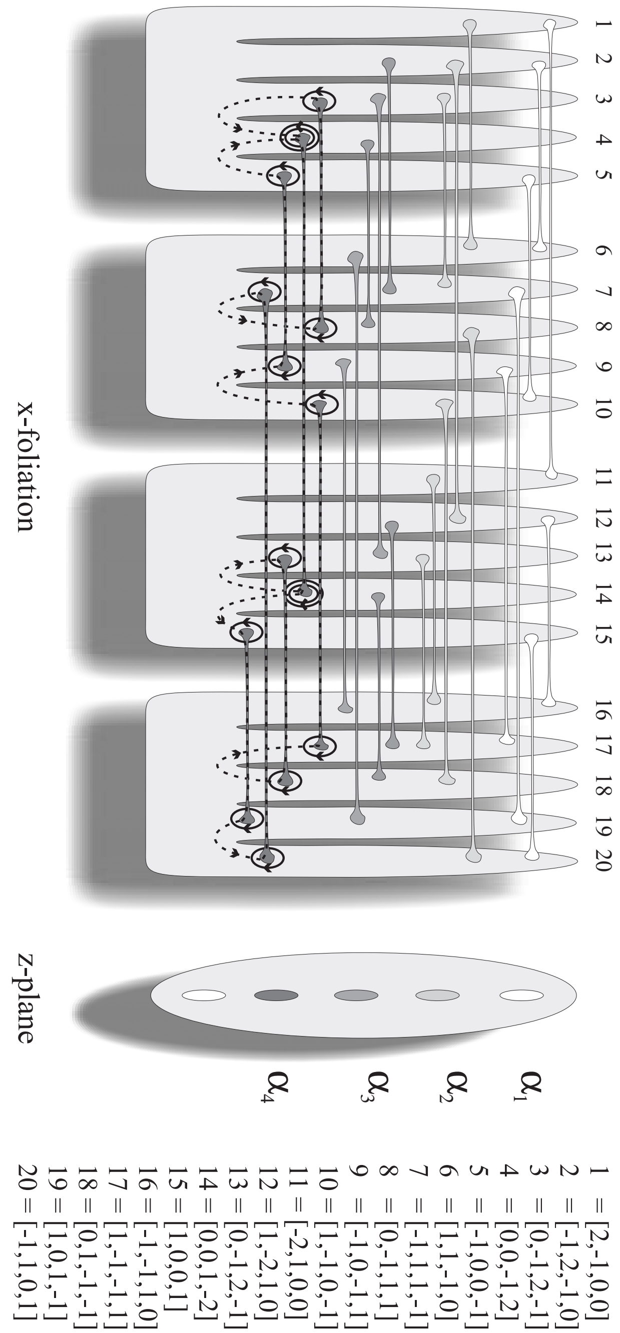

Figure 3.6: *The Riemann surface for A_4 in the 24-dimensional adjoint representation, including the cycles above the fourth simple root. The fourth root and fourth weight are equal and their norm is two, thus causing two cycles of type A to encircle that branch cut and two strands to go up to the branch cut to form a special B cycle. The special A cycle is therefore obtained by adding all type A cycles in the picture and the special B cycle by adding the B type cycles, denoted by dotted lines.*

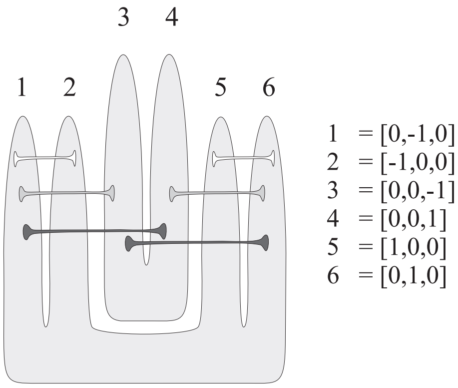

Figure 3.7: *The curve for D_3 in the smallest 6-dimensional representation. Sending all weights ω_i to $-\omega_i$ is an involution of the curve, the same involution as sending $x \to -x$.*

Now consider the bilinear form $\sum_\omega (\omega \cdot x)(\omega \cdot y)$ on the linear space where the roots take their values. This bilinear form is invariant under the Weyl group and therefore we find that equals a multiple of the Euclidean inner product on the root space. So in the end we find that

$$A_i \circ B_j \sim \frac{1}{\alpha_i \cdot \alpha_i}\delta_{ij} \sim \frac{1}{2}\delta_{ij} \tag{3.1.46}$$

which is a multiple of the identity as it should be.

The special cycles for curves of D_N have been defined in two ways, which we show to be identical. The weights come in pairs $\omega_i, -\omega_i$ which are related through $x \to -x$ because of (3.1.24). Every weight has nonzero inner product with only one simple root, so that the A cycles (3.1.38) and (3.7) and the corresponding B cycles are the same. See figure 3.7 for more details.

3.2 Definition of the prepotential

The Seiberg-Witten data has been introduced, consisting of the family of curves $\Sigma_{\mathfrak{g},\rho}$ (definition 3.3), the Seiberg-Witten differential λ_{SW} (definition 3.5) and a canonical subset of $2N$ cycles A_i and B_j with the usual intersection numbers (definitions 3.7 and 3.8). We will need the following lemma

Lemma 3.9. *There exists an additional set of cycles $A_{N+1}, ..., A_g$ with the appropriate intersection numbers with the special cycles, and with the property that the period integrals of λ_{SW} around them are zero.*

In particular, this lemma implies that the special cycles are a subset of a canonical homology basis.

Proof. For the classical Lie algebras, the additional A cycles are given by the σ-invariant combinations

$$\tilde{A}_i = C_i^+ + C_-^- \tag{3.2.1}$$

see also (3.1.38). Since the period integrals are independent of x, we find that σ acts as the identity on them. On the other hand, the involution σ sends λ_{SW} to $-\lambda_{SW}$ and therefore we conclude

$$\sigma\left(\oint_{\tilde{A}_i}\lambda_{SW}\right) = -\oint_{\tilde{A}_i}\lambda_{SW} = 0 \tag{3.2.2}$$

Regarding the non simply laced Lie algebras, we again refer to [29] for the details about the special cycles. For the simply laced ones, there is a special cycle A_i for each root α_i. After our construction of additional cycles, the number of A cycles equals the number of branch cuts for finite values of x. These are too many cycles since the genus is the number of branch cuts minus the number of cuts necessary to connect the different Coxeter orbits of weights. Selecting a subset with g elements (including the special cycles) gives the set of A cycles promised by the lemma.

Take a simple root α_i. There are just as many branch cuts above α_i as there are weights ω with $\alpha_i \cdot \omega > 0$. Corresponding to α_i, take a weight ω_i so that $\omega_i \cdot \alpha_i > 0$. We introduce the subset Ω_i of the set of weights Ω by

$$\Omega_i = \{\omega' \in \Omega | (\omega' \cdot \alpha_i) > 0\ ,\ \omega' \neq \omega_i\} \tag{3.2.3}$$

For every $\omega'_k \in \Omega_i$ we define the cycle

$$A_i(\omega'_k) = \hat{A}_i^{\omega_i} - \frac{\alpha_i \cdot \omega_i}{\alpha_i \cdot \omega'_k}\hat{A}_i^{\omega'_k} \tag{3.2.4}$$

where $\hat{A}_i^{\omega_i}$ is defined in section 3.1.5. Together with the special cycle A_i this gives a number of cycles for each simple root α_i equal to the number of branch cuts for α_i.

We calculate the intersection numbers with the B_j and find

$$\begin{aligned} A_i(\omega'_k) \circ B_j = N_{j,\rho}\left(\hat{A}_i^{\omega_i} - \frac{\alpha_i \cdot \omega_i}{\alpha_i \cdot \omega'_k}\hat{A}_i^{\omega'_k}\right) \circ \sum_{\omega''}\alpha_i \cdot \omega'' B_j^{\omega''} = \\ N_{i,\rho}\left(\alpha_i \cdot \omega_i - \frac{\alpha_i \cdot \omega_i}{\alpha_i \cdot \omega'_k}\alpha_i \cdot \omega'_k\right)\delta_{ij} = 0 \end{aligned} \tag{3.2.5}$$

Moreover, we will show that the period integrals of λ_{SW} over the cycles $A_i(\omega'_k)$ are zero:

$$\begin{aligned} \oint_{A_i(\omega'_k)}\lambda_{SW} = \oint_{\hat{A}_i^{\omega_i}}\lambda_{SW} - \frac{\alpha_i \cdot \omega_i}{\alpha_i \cdot \omega'_k}\oint_{\hat{A}_i^{\omega'_k}}\lambda_{SW} = \\ -\frac{\alpha_i \cdot \omega_i}{\alpha_i \cdot \alpha_i}\oint_{C_i}\alpha_i \cdot v(z)\frac{dz}{z} + \frac{\alpha_i \cdot \omega_i}{\alpha_i \cdot \omega'_k}\frac{\alpha_i \cdot \omega'_k}{\alpha_i \cdot \alpha_i}\oint_{C_i}\alpha_i \cdot v(z)\frac{dz}{z} = 0 \end{aligned} \tag{3.2.6}$$

Repeating this construction of cycles for each simple root, we find that the number of A cycles now equals the number of branch cuts for finite values of x. As mentioned before, these are too many since some cycles are needed to connect the different Coxeter orbits of weights. We can always make a selection such that the cycles that are left out connect the Coxeter orbits. Thus we end up with a set of g canonical A cycles promised by the lemma. □

Using lemma 3.9 in combination with proposition 3.6, we can define the prepotential. First we define the new variables

$$a_i = \oint_{A_i} \lambda_{SW} \tag{3.2.7}$$

which we can use to make a local change of variables on the moduli space. To prove that the change of variables from u_i to a_i is nonsingular, we note that the integrals of the holomorphic differentials $\frac{\partial \lambda_{SW}}{\partial u_i}$ around the cycles $A_{N+1}, ..., A_g$ are zero. Since the N by g matrix

$$\oint_{A_j} \frac{\partial \lambda_{SW}}{\partial u_i} \tag{3.2.8}$$

must have rank N, we conclude that the determinant of the Jacobi matrix for the change of variables from u_i to a_i is nonzero.

This is similar to the situation for Lie algebra A_N, which we discussed in section 3.1.1. We proceed to define the b_j by

$$b_j = \oint_{B_j} \lambda_{SW} \tag{3.2.9}$$

and their moduli derivatives

$$\frac{\partial b_j}{\partial a_i} = \Pi_{ij} \tag{3.2.10}$$

Since the special A cycles are a subset of a canonical homology basis and since the holomorphic forms $\frac{\partial \lambda_{SW}}{\partial a_i}$ are canonical with respect to this basis

$$\oint_{A_i} \frac{\partial \lambda_{SW}}{\partial a_j} = \delta_{ij} \qquad 1 \leq i \leq g \qquad 1 \leq j \leq N \tag{3.2.11}$$

we find that Π_{ij} is an N by N submatrix of the g by g period matrix, and therefore symmetric. Due to this symmetry we can locally integrate the b_j and find the prepotential $\mathcal{F}$.

Definition 3.10. *The prepotential $\mathcal{F}(a_1, ..., a_N)$ is defined locally on the moduli space by*

$$b_j = \frac{\partial \mathcal{F}}{\partial a_j} \tag{3.2.12}$$

In section 3.1.5 it was shown that a_i and b_j are representation independent, which shows that although we have chosen the smallest representation to define the family of curves we could in fact have used any irreducible representation and the prepotential is independent of this choice.

The prepotential is not independent however of the choice of the special cycles. In particular, a symplectic change of these cycles results in the definition of a different prepotential. Such a symplectic change of cycles has the same effect as a symplectic transformation as discussed in section 1.2.4, where it was shown that such transformations are contact symmetries of the WDVV equations. Therefore, the different prepotentials either simultaneously satisfy the WDVV equations or they all don't. In the following section a proof is given that the prepotentials do satisfy the WDVV equations, and this proof does not depend on the choice of cycles.

3.3 The WDVV equations

In this section we prove that the prepotentials $\mathcal{F}(a_1,...,a_N)$, introduced in definition 3.10, satisfy the WDVV system (1.2.2). To do this, we construct in section 3.3.2 a family of associative commutative algebras with structure constants $C_{ij}^k(a)$ depending on the moduli a_i and we relate these structure constants with the third order derivatives of $\mathcal{F}$ through (1.2.3)

$$\begin{aligned} \mathcal{F}_{ijk} &= \sum_{l,m=1}^{N} C_{ij}^l(a) K_{kl} \\ K_{kl} &= \alpha_m \mathcal{F}_{klm} \end{aligned} \tag{3.3.1}$$

for some set of α_m, possibly a_i dependent. As explained in section 1.2, this proves that the prepotential $\mathcal{F}$ satisfies the WDVV system. We will use two different methods to prove the relation 3.3.1: the first method [32] uses Picard-Fuchs equations and flat coordinates, and the second method [47] uses a more widely applicable residue formula.

3.3.1 Preliminaries: Term orderings and Groebner bases

As preparation for the definition of the associative commutative algebras, we discuss some basic aspects of the theory of ideals in polynomial rings, see e.g. [11]. For polynomial rings $\mathbf{C}[x]$ in one variable, ideals I are always generated by a single element. This generator is up to a constant uniquely identified as the element of the ideal with minimal degree in x. To determine whether a polynomial is in the ideal or not we divide this polynomial by the generator. If there is a zero remainder the polynomial is in I, otherwise not.

For polynomial rings in two or more variables the situation is more difficult. It can be shown that every ideal is finitely generated, but the number of generators usually exceeds one. Also, division by the generators has become less clear: in $\mathbf{C}[x]$ one divides by looking at the highest degree term in x and the rest simply follows. Here it is not clear which term has highest degree. To fix this one introduces a term ordering, a total ordering which prescribes what is the leading term of a polynomial. For instance, the lexicographical ordering in $\mathbf{C}[x,y]$ sais that one should first look at the powers of x occurring in the polynomial and if there are equal powers then further distinction is made using the powers of y. As an example we consider $x^4 + y^2x^2 + y^2x + yx$ whose leading term is x^4 in the lexicographical term ordering.

Now that we have introduced the term ordering, we can divide polynomials by the ideal generators to determine whether or not they are in I. However, the order of division influences the outcome: the remainder after several divisions can contain different representatives of the same equivalence class in $\mathbf{C}[x,y]$ depending on the order of division. A *Groebner basis* of generators for the ideal is a particular basis with two special properties: the first one is that the order of division is irrelevant, the outcome is always the same. The second property is that an element of the ideal gives zero remainder after division regardless of the term ordering. After the construction of a Groebner basis, membership of the ideal can therefore be decided using a straightforward division algorithm.

We will now briefly describe Buchberger's algorithm [11] to obtain a Groebner basis from a given set of generators $p_1,...,p_n$. First one defines the S-polynomial $S(p_1,p_2)$ of two polynomials. Multiply p_1 and p_2 with monomials of minimal degree (with respect to the term ordering) such that their leading terms become equal. Then subtract one from the other and this gives $S(p_1, p_2)$. For instance, in the lexicographical term ordering we have

$$\begin{aligned} S(x^4 + y^2x^2 + y^2x + yx, yx^2 + y^3x) = \\ y\left(y^2x^2 + y^2x + yx + x^4\right) - x^2\left(y^3x + yx^2\right) = \\ -y^3x^3 + y^3x^2 + y^3x + y^2x \end{aligned} \quad (3.3.2)$$

The algorithm to produce a Groebner basis is now as follows: first one takes the basis $p_1, ..., p_n$ and divides the polynomials amongst each other in random order. If a division is possible then we replace that polynomial by its remainder. Then we add the S-polynomial of two random elements in the basis and divide it in random order by the other basis elements, again replacing it by its remainder if division is possible. We repeat this process over and over again, until every S-polynomial of two polynomials is already in the basis. We have then obtained a Groebner basis.

3.3.2 The family of associative algebras

For each simple Lie algebra, we will construct a family of polynomial algebras over an ideal. Since they are polynomial, they are automatically commutative and associative. Furthermore, the choice of a unit element will eventually determine the precise linear combination K appearing in the WDVV equations (1.2.2).

We will again denote the algebraic curves by

$$P(x, z + \frac{\mu}{z}, u_i) = 0 \quad (3.3.3)$$

and for our convenience we will consider them as the double cover of a torus

$$\begin{aligned} P(x, w, u_i) = 0 \\ z + \frac{\mu}{z} = w \end{aligned} \quad (3.3.4)$$

The function P is now a polynomial in the two variables x, w. We introduce the ideal $I = \langle P, P_x \rangle$ in $\mathbf{C}[x, w]$. We will check for each simply laced Lie algebra that the $P_{a_i} = \frac{\partial P}{\partial a_i}$ span a subalgebra of $\mathbf{C}[x, w]/I$.

Definition 3.11. *For any simple Lie algebra $\mathfrak{g}$ whose family of Seiberg-Witten curves is given by $P(x, w) = 0$, the family of algebras $\mathcal{A}$ is defined by taking subalgebras of $\mathbf{C}[x, w]/I$ where I is the ideal generated by P and P_x. These subalgebras are the ones generated by the P_{a_i} and are automatically associative and commutative as subalgebras of a polynomial algebra.*

Since the Seiberg-Witten family of curves is formulated in terms of the u_i as moduli, we will give often give the algebras in terms of the $P_{u_j} = \sum_j \frac{\partial a_i}{\partial u_j} P_{a_i}$ which span the same subalgebra as the P_{a_i}. In fact, given any good local coordinate system on the moduli space, the algebra can always be defined in terms of the derivatives of P with respect to those coordinates. In terms of the u_j the structure constants are defined through

$$P_{u_i} P_{u_j} = \sum_{k,q} C_{ij}^k(\alpha, u) P_{u_k} \alpha_q P_{u_q} \qquad \mod I \quad (3.3.5)$$

where $\sum_q \alpha_q P_{u_q}$ serves as the unit element of the algebra. The dependence of the structure constants on the unit element and the coordinates u_j is emphasized. Making the change of coordinates to the a_i we find that the structure constants transform as a $(2,1)$ tensor into

$$C_{ij}^k(\beta, a) = \sum_{l,m,n} \frac{\partial u_l}{\partial a_i} \frac{\partial u_m}{\partial a_j} C_{lm}^n(\alpha, u) \frac{\partial a_k}{\partial u_n} \tag{3.3.6}$$

and the new algebra unit is

$$\sum_p \beta_p P_{a_p} = \sum_{p,q} \alpha_q \frac{\partial a_p}{\partial u_q} P_{a_p} \tag{3.3.7}$$

3.3.2.1 Three realizations of the same algebra

It will be useful to have three realizations of the same algebra: to prove that it exists, we will use the polynomial multiplication

$$P_{u_i} P_{u_j} = \sum_{k,q} C_{ij}^k P_{u_k} \alpha_q P_{u_q} + \bar{Q}_{ij} P_x \tag{3.3.8}$$

To make the connection with flat coordinates and Landau-Ginzburg theory in section 3.3.3, we will use an algebra of rational functions whose multiplication reads

$$w_{u_i} w_{u_j} = \sum_{k,q} C_{ij}^k w_{u_k} \alpha_q w_{u_q} + \frac{-\bar{Q}_{ij}}{P_w} w_x = \sum_{k,q} C_{ij}^k w_{u_k} \alpha_q w_{u_q} + Q_{ij} w_x \tag{3.3.9}$$

where $w_{u_i} = -\frac{P_{u_i}}{P_w}$ and w plays the role of a one-variable Landau-Ginzburg superpotential. Finally, to show in subsection 3.3.7 that the algebraic function $w(x, u_i)$ is a superpotential for any choice of the representation, we will regard the algebra as an algebra of holomorphic forms [47]

$$\frac{\partial \lambda_{SW}}{\partial u_i} \otimes \frac{\partial \lambda_{SW}}{\partial u_j} = \sum_{k,q} C_{ij}^k \frac{\partial \lambda_{SW}}{\partial u_k} \otimes \alpha_q \frac{\partial \lambda_{SW}}{\partial u_q} + \frac{\bar{Q}_{ij}}{P_x} \frac{dz}{z} \otimes \frac{dz}{z} \tag{3.3.10}$$

The elements of the left and right hand sides of this equation are elements of Ω^2, the space of holomorphic quadratic differentials.

But first we will prove the existence of the algebras in the upcoming paragraphs, using the polynomial algebra (3.3.8).

Type $\mathbf{A_N}$

The family of Riemann surfaces in this case is given by

$$P_{A_N}(x, w) = w + W(x, u_i) = 0 \tag{3.3.11}$$

where W is the A_N Landau-Ginzburg superpotential. The ideal $I \subset \mathbf{C}[x, w]$ is given by $I = \langle w + W, W_x \rangle$. Since $P_{u_i} = W_{u_i}$ depends only on x we find that we can restrict our attention to $\mathbf{C}[x]/J$ where J is the ideal generated by W_x.

Consider the polynomial algebra

$$P_{u_i} P_{u_j} = \sum_{k,m=1}^{N} C_{ij}^{k} P_{u_k} \alpha_m P_{u_m} \qquad \text{mod } P_x \tag{3.3.12}$$

Due to the particular P under consideration, this algebra simplifies to

$$W_{u_i} W_{u_j} = \sum_{k,q=1}^{N} C_{ij}^{k}(\alpha, u) W_{u_k} \alpha_q W_{u_q} \qquad \text{mod } W_x \tag{3.3.13}$$

which is just the well-known Landau-Ginzburg algebra. The algebra exists because W_x is a degree N polynomial in x generating the ideal J in $\mathbf{C}[x]$ and the $W_{u_i} = x^{N-i}$ form a basis of $\mathbf{C}[x]/J$. Since it is a polynomial algebra, it is automatically associative and commutative. As an example, we give the structure constants $C_{ij}^{k}(\alpha_q = \delta_{q,4}, u)$ of A_4.

$$(C_1)_j^k = \begin{pmatrix} -\frac{2}{5}u_2 & -\frac{1}{5}u_2 + \frac{9}{25}u_1^2 & \frac{6}{25}u_1 u_2 & \frac{3}{25}u_1 u_3 \\ -\frac{3}{5}u_1 & -\frac{2}{5}u_2 & -\frac{1}{5}u_3 & 0 \\ 0 & -\frac{3}{5}u_1 & -\frac{2}{5}u_2 & -\frac{1}{5}u_3 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.3.14}$$

$$(C_2)_j^k = \begin{pmatrix} -\frac{3}{5}u_1 & -\frac{2}{5}u_2 & -\frac{1}{5}u_3 & 0 \\ 0 & -\frac{3}{5}u_1 & -\frac{2}{5}u_2 & -\frac{1}{5}u_3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$(C_3)_j^k = \begin{pmatrix} 0 & -\frac{3}{5}u_1 & -\frac{2}{5}u_2 & -\frac{1}{5}u_3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$(C_4)_j^k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Type $\mathbf{B_N}$

The family of Riemann surfaces in this case is given by

$$P_{B_N}(x,w) = xw + W^{BC} = xw + x^{2N} + u_1 x^{2N-2} + u_2 x^{2N-4} ... + u_N = 0 \quad (3.3.15)$$

where W^{BC} is the type BC Landau-Ginzburg superpotential. The ideal I is given by $I = \langle xw + W^{BC}, w + W_x^{BC} \rangle$. Since $P_{u_i} = W_{u_i}^{BC}$ depends only on x we find that we can restrict our attention to $\mathbf{C}[x]/J$ with an ideal J. To see what J should be, we calculate a Groebner basis of I in terms of a lexicographical order in which $w > x$ and we find that the only element in the basis not depending on w is $W^{BC} - xW_x^{BC}$. To see that this is an element of I we note that

$$W^{BC} - xW_x^{BC} = \left(xw + W^{BC}\right) - x\left(w + W_x^{BC}\right) \quad (3.3.16)$$

The quotient ring $\mathbf{C}[x]/J$ consists of polynomials up to degree $2N$ and since $W^{BC} - xW_x^{BC}$ contains only even degree terms the P_{u_k} span a subalgebra consisting of polynomials of even degree in $\mathbf{C}[x]/J$.

As an example, we give the structure constants $C_{ij}^k(\alpha_q = \delta_{q,3}, u)$ of the algebra for B_3 and note that this is not the Landau-Ginzburg algebra of type BC [64]. This is no coincidence: due to the twisting procedure from $\mathfrak{g}^{(1)}$ to $\left(\mathfrak{g}^{(1)}\right)^\vee$ in the definition of the Seiberg-Witten family of curves, the relationship between the Seiberg-Witten algebra and the Landau-Ginzburg algebra is lost for the non simply laced Lie algebras. For the simply laced ones the two algebras are in fact the same, as we will see.

$$
\begin{aligned}
(C_1)_j^k &= \begin{pmatrix} -\frac{1}{5}u_2 + \frac{9}{25}u_1^2 & \frac{1}{5}u_3 + \frac{3}{25}u_1u_2 & -\frac{3}{25}u_1u_3 \\ & & \\ -\frac{3}{5}u_1 & -\frac{1}{5}u_2 & \frac{1}{5}u_3 \\ & & \\ 1 & 0 & 0 \end{pmatrix} \\
(C_2)_j^k &= \begin{pmatrix} -\frac{3}{5}u_1 & -\frac{1}{5}u_2 & \frac{1}{5}u_3 \\ & & \\ 1 & 0 & 0 \\ & & \\ 0 & 1 & 0 \end{pmatrix} \\
(C_3)_j^k &= \begin{pmatrix} 1 & 0 & 0 \\ & & \\ 0 & 1 & 0 \\ & & \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Type $\mathbf{C_N}$

The family of Riemann surfaces in this case is given by

$$P_{C_N}(x,w) = w^2 - 4\mu + x^2 W^{BC} = 0 \tag{3.3.17}$$

The ideal I is given by $I = \langle w^2 - 4\mu + x^2 W^{BC}, 2xW^{BC} + x^2 W_x^{BC} \rangle$. Since P_{u_i} depends only on x we find that we can restrict our attention to $\mathbf{C}[x]/J$ with an ideal J where J is generated by $2xW^{BC} + x^2 W_x^{BC}$. The quotient ring $\mathbf{C}[x]/J$ consists of polynomials up to degree $2N+1$ and since $2xW^{BC} + x^2 W_x^{BC}$ contains only odd degree terms the polynomials of even degree span a subalgebra in $\mathbf{C}[x]/J$. The dimension of this subalgebra however is $N+1$, and we only have N polynomials P_{u_i}. Still the P_{u_i} which have degree in x greater or equal to 2 span yet a smaller subalgebra, because the lowest degree in x occurring in the ideal generator is degree 3.

Type $\mathbf{D_N}$

The family of Riemann surfaces in this case is given by

$$
\begin{aligned}
P_{D_N}(x,w) = x^2 w + W^D &= \\
& x^2 w + x^{2N} + u_1 x^{2N-2} + ... + u_{N-2}x^4 + u_N x^2 + u_{N-1}^2 = 0
\end{aligned} \tag{3.3.18}
$$

The ideal I is given by $I = \langle x^2 w + W^D, 2xw + W_x^D \rangle$. Since $P_{u_i} = W_{u_i}^D$ depends only on x we find that we can restrict our attention to $\mathbf{C}[x]/J$ with an ideal J. To see what J should be, we calculate a Groebner basis of I in terms of a lexicographical order in which $w > x$ and we find that the only element in the basis not depending on w is $2W^D - xW_x^D$. To see that this is an element of I we note that

$$2W^D - xW_x^D = 2\left(x^2 w + W^D\right) - x\left(2xw + W_x^D\right) \tag{3.3.19}$$

The quotient ring $\mathbf{C}[x]/J$ consists of polynomials up to degree $2N$ and since $2W^D - xW_x^D$ contains only even degree terms the P_{u_k} span a subalgebra consisting of polynomials of even degree in $\mathbf{C}[x]/J$. Note that this is precisely the Landau-Ginzburg algebra for type D_N.

Type $\mathbf{E_6}$

Until now, the polynomial $P(x, w, u_i)$ did not contain terms mixing w with the moduli u_i. This allowed us to consider polynomial algebras in one variable. Any ideal is then generated by just one polynomial and calculations are done by dividing by this polynomial. For E_6 this is no longer the case. Since mixing does occur, we are forced to use the two-variable ring $\mathbf{C}[x, w]$ in which it is no longer guaranteed that an ideal is generated by one polynomial. Nevertheless one can construct a finite Groebner basis for the ideal in such a way that calculations in the quotient ring can be done by using a division algorithm to divide out the elements of the basis.

An additional help in explicit computations is the grading that is present. As mentioned before, the principal grading of the affine Lie algebra causes the Riemann surfaces and Seiberg-Witten differential to be graded as well, and in turn the algebra that we are constructing is graded. Since the dependence on the Casimirs u_i is always polynomial, we can predict the dependence of the structure constants $C_{ij}^k(u)$ on the Casimirs. The only thing we have to calculate explicitly are the coefficients of the various terms, which are just numbers. For example, if we take $\alpha_q = \delta_{q,6}$ then the algebra becomes

$$P_{u_i} P_{u_j} = \sum_k C_{ij}^k(u) P_{u_k} P_{u_6} \bmod I \tag{3.3.20}$$

The degree of P is 27, the degrees of the Casimirs $u_1, ..., u_6$ are respectively $2, 5, 6, 8, 9, 12$ and thus $C_{12}^3(u)$ for example has degree 11. The terms that constitute C_{12}^3 are therefore $u_1^3 u_2$, $u_2 u_3$ and $u_1 u_5$ and only their coefficients need to be determined.

Explicit computation of the Groebner basis (using a lexicographical term ordering) shows that the quotient algebra $\mathbf{C}[x, w]/I$ is 57-dimensional, and the algebra generated by the P_{u_i} is a 6-dimensional subalgebra. The fact that it's a closed subalgebra is by no means trivial. This subalgebra is precisely the Landau-Ginzburg algebra [21].

Type $\mathbf{F_4}$

Again we have used Groebner bases theory together with the grading to determine the structure constants. Explicit computation of the Groebner basis (using a lexicographical term ordering) shows that the quotient algebra $\mathbf{C}[x, w]/I$ is 78-dimensional, and the algebra generated by the P_{u_i} is a nontrivial 4-dimensional subalgebra. Just like in the other non simply laced cases this is not the Landau-Ginzburg algebra of type F_4, which is given in [64]. The structure constants $C_{ij}^k(\alpha_q = \delta_{q,4}, u)$ are given by[4] [26]:

$$\left(C_1^T\right)_j^k = \begin{pmatrix} u_1\left(\frac{250}{243}u_1^4 - \frac{10}{9}u_1u_2 - \frac{7}{3}u_3\right) & -\frac{25}{54}u_1^3 + \frac{1}{4}u_2 & -\frac{5}{3}u_1^2 & 1 \\ \frac{100}{81}u_1^4u_2 + \frac{140}{27}u_1^3u_3 - \frac{2}{3}u_1u_2^2 - \frac{4}{3}u_1u_4 - 2u_2u_3 & u_1\left(-\frac{5}{9}u_1u_2 - \frac{7}{3}u_3\right) & -6u_3 - 2u_1u_2 & 0 \\ -\frac{2}{9}u_1u_2u_3 - \frac{2}{3}u_3^2 + \frac{100}{243}u_1^4u_3 - \frac{10}{27}u_1^2u_4 & \frac{1}{6}u_4 - \frac{5}{27}u_1^2u_3 & -\frac{2}{3}u_1u_3 & 0 \\ \frac{10}{9}u_1^2u_3^2 - \frac{1}{3}u_1u_2u_4 - u_3u_4 + \frac{50}{81}u_1^4u_4 & -\frac{1}{2}u_3^2 - \frac{5}{18}u_1^2u_4 & -u_1u_4 & 0 \end{pmatrix}$$

$$\left(C_2^T\right)_j^k = \begin{pmatrix} -\frac{25}{54}u_1^3 + \frac{1}{4}u_2 & \frac{5}{24}u_1 & \frac{3}{4} & 0 \\ u_1\left(-\frac{5}{9}u_1u_2 - \frac{7}{3}u_3\right) & \frac{1}{4}u_2 & 0 & 1 \\ \frac{1}{6}u_4 - \frac{5}{27}u_1^2u_3 & \frac{1}{12}u_3 & 0 & 0 \\ -\frac{1}{2}u_3^2 - \frac{5}{18}u_1^2u_4 & \frac{1}{8}u_4 & 0 & 0 \end{pmatrix}$$

$$\left(C_3^T\right)_j^k = \begin{pmatrix} -\frac{5}{3}u_1^2 & \frac{3}{4} & 0 & 0 \\ -6u_3 - 2u_1u_2 & 0 & -6u_1 & 0 \\ -\frac{2}{3}u_1u_3 & 0 & 0 & 1 \\ -u_1u_4 & 0 & -\frac{9}{2}u_3 & 0 \end{pmatrix}$$

4 To get a better lay-out, we give the transpose matrices $\left(C_i^T\right)_j^k = (C_i)_k^j$.

$$\left(C_4^T\right)_j^k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It can be checked explicitly that these are indeed the structure constants of an associative commutative algebra.

Type $\mathbf{G_2}$

Finally, we arrive at the G_2 case. Although the WDVV equations are trivially satisfied, we give the family of associative algebras to show how it fits the general pattern. Since the Groebner basis of the ideal generated by P and P_x is not so big, we can give it explicitly:

$$\Big\{ \quad 288u^2x^9 + 192x^{13} - 384ux^{11} - 1728x^5\mu - 12u^2x^3v - 48u^2x\mu + \\ 24uvx^5 + 576u\mu x^3 + 16u^4x^5 + 3xv^2 - 112x^7u^3 + 48x^7v, \\ -288x^{11} + 528ux^9 - 344u^2x^7 - 90vx^5 + 2592\mu x^3 - \\ 54vxw - 432xu\mu + 114x^5u^3 - 24x^3uv - 10u^4x^3 + 5u^2xv + 10u^3xw, \\ -162vw^2 + 30u^3w^2 + 288x^{12}u - 528x^{10}u^2 - \\ 54vx^8 + 354u^3x^8 - 124u^4x^6 + 144vux^6 - 2592ux^4\mu + \\ 24u^2x^4v + 10u^5x^4 + 432u^2x^2\mu - 27x^2v^2 + 648v\mu - 120u^3\mu \Big\} \tag{3.3.21}$$

The resulting structure constants with $\alpha_q = \delta_{q,2}$ are

$$\begin{aligned} C_1 &= \begin{pmatrix} -\frac{2}{3}u^2 & -\frac{2}{3}uv + 16\mu \\ 1 & 0 \end{pmatrix} \\ C_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Lie algebra G_2 constitutes the only example where due to the twisting procedure the parameter μ appears explicitly in the structure constants, making it painstakingly clear that the direct relation between this algebra and the Landau-Ginzburg algebra is lost.

After having introduced the prepotential $\mathcal{F}$ and family of algebras $\mathcal{A}$ separately, it remains to relate the two. There are two methods known in the literature of doing this. One method exploits the existence of flat coordinates in the Landau-Ginzburg context and interprets the relation (3.3.1) as Picard-Fuchs equations [32]. It has the drawback of not being directly applicable to the non simply laced Lie algebras, for which flat coordinates in general do not exist. The other method is more widely applicable and uses a residue formula [47], [39]. We will explain both methods in detail below.

3.3.3 The Gauss-Manin connection, flat coordinates and Picard-Fuchs equations

This section deals only with the simply laced Lie algebras, since there is a natural connection between the structure constants of the algebra and the definition of flat coordinates for them. The non simply laced algebras are discussed in the next section.

Given a family of subvarieties $X \subset \mathbf{P}^n$ fibered over a moduli space $\mathcal{M}$, there is a method dating back to Griffiths [23] of obtaining a set of differential equations for period integrals when differentiated with respect to the moduli. Such equations are called Picard-Fuchs equations. Let X be given by an affine equation $P(x_1, ..., x_n) = 0$ and take a closed cycle $\Xi \subset \mathbf{P}^n$ which encloses X. We consider integrals of the type

$$\zeta^{(l)} = \int_\Xi \frac{\phi}{P^l}\Omega \tag{3.3.22}$$

where ϕ is a polynomial and Ω is the form on $\mathbf{P}^n$ given in local coordinates by

$$\Omega = dx_1 \wedge ... \wedge dx_n \tag{3.3.23}$$

in the coordinate patch where $x_{n+1} \neq 0$. Differentiating $\zeta^{(l)}$ with respect to the moduli, we get

$$\frac{\partial \zeta^{(l)}}{\partial u_j} = \int_\Xi \left(\frac{\frac{\partial \phi}{\partial u_j}}{P^l} - l\frac{\phi\frac{\partial P}{\partial u_j}}{P^{l+1}} \right)\Omega \tag{3.3.24}$$

The main idea is to perform a series of partial integrations to reduce the powers of P occurring in the denominator: each term of the form

$$\int_\Xi l\frac{\psi\frac{\partial P}{\partial x_k}}{P^{l+1}}\Omega \tag{3.3.25}$$

equals

$$\pm \int_\Xi d\left(\frac{\psi}{P^l} dx_1 \wedge ... \wedge \widehat{dx_k} \wedge ... \wedge dx_n\right) \mp \int_\Xi \frac{\frac{\partial \psi}{\partial u_k}}{P^l}\Omega \tag{3.3.26}$$

So we have to divide $\phi\frac{\partial P}{\partial u_j}$ by the various $\frac{\partial P}{\partial x_k}$ in order to do those partial integrations. By chosing a term ordering and constructing a Groebner basis for the ideal I generated by the $\frac{\partial P}{\partial x_k}$ one makes sure that the order of division is irrelevant.

In case X is a miniversal deformation of a singularity of ADE type [5], $\mathbf{C}[x_1, ..., x_n]/I$ is called the Jacobian ring and its dimension the Milnor number of the singularity. The $\frac{\partial P}{\partial u_i}$ generate a finite-dimensional subalgebra of the Jacobian ring and one can consider the integrals

$$\zeta_i^{(l)} = \int_\Xi \frac{\frac{\partial P}{\partial u_i}}{P^l}\Omega \tag{3.3.27}$$

Using the algebra together with the partial integrations one gets the following set of differential equations

$$\frac{\partial \zeta_i^{(l)}}{\partial u_j} - lC_{ij}^k \zeta_k^{(l+1)} + \sum_n \Gamma_{ij}^{(n)k}\zeta_k^{(l-n)} = 0 \tag{3.3.28}$$

More formally this is the equation of a flat connection, called the Gauss-Manin connection, on a cohomology bundle over $\mathcal{M}$ of which the $\zeta_i^{(l)}$ are sections. One can check the integrability conditions of the connection separately for each power of P in the denominator, which leads to the following identities on the structure constants

$$\begin{aligned} [C_i, C_j] &= 0 \\ \frac{\partial C_{ij}^k}{\partial u_l} &= \frac{\partial C_{lj}^k}{\partial u_i} \end{aligned} \tag{3.3.29}$$

where C_i is the matrix with coefficients C_{ij}^k. The first of these equations expresses the associativity of the algebra, and is automatically fulfilled. The second puts an integrability condition on the structure constants, so that $C_{ij}^k = \frac{\partial^2 T^k}{\partial u_i \partial u_j}$ for some set of functions T^k. Saito [58] then goes on to construct the flat coordinates, in terms of which the connection $\Gamma_{ij}^{(0)k}$ vanishes.

As an alternative to the integrals over Ξ, we can use the higher dimensional analogue of Cauchy's residue theorem [10] to study period integrals over closed cycles on X itself, on which $P = 0$. We will consider the family of Riemann surfaces Σ as subvarieties of $\mathbf{P}^2$ fibered over $\mathcal{M}$. We have indicated in section 3.1.1 how to differentiate cohomology elements with respect to the moduli. We consider the subring B of the full cohomology ring, generated by $\frac{\partial \lambda_{SW}}{\partial u_i}$ and $\frac{\partial^2 \lambda_{SW}}{\partial u_i \partial u_j}$ with $i \leq j$. It is not hard to see that these are all linearly independent and therefore constitute a basis $\{\chi_i\}$ of the subring B. We will need the following lemma

Lemma 3.12. *[21, 31] For simply laced Lie algebras, the following Picard-Fuchs equations hold in the cohomology subring B*

$$\frac{\partial^2 \lambda_{SW}}{\partial u_i \partial u_j} - \sum_k C_{ij}^k(u) \frac{\partial^2 \lambda_{SW}}{\partial u_k \partial u_N} + \frac{\frac{\partial w}{\partial u_i \partial u_j} - \frac{\partial Q_{ij}}{\partial x}}{\sqrt{w^2 - 4\mu}} dx = 0 \tag{3.3.30}$$

where the structure constants $C_{ij}^k(u)$ are defined through (3.3.5), using $\alpha_q = \delta_{q,N}$.

Proof. Using $w = z + \frac{\mu}{z}$, the first order derivative of λ_{SW} equals

$$\frac{\partial \lambda_{SW}}{\partial u_i} = \frac{\partial \log(z)}{\partial u_i} dx = \frac{1}{z} \frac{dz}{dw} \frac{\partial w}{\partial u_i} dx = \frac{\frac{\partial w}{\partial u_i}}{\sqrt{w^2 - 4\mu}} dx \tag{3.3.31}$$

and therefore the second order derivative equals

$$\frac{\partial^2 \lambda_{SW}}{\partial u_i \partial u_j} = \frac{\frac{\partial w}{\partial u_i \partial u_j}}{\sqrt{w^2 - 4\mu}} dx - \frac{w \frac{\partial w}{\partial u_i} \frac{\partial w}{\partial u_j}}{\left(\sqrt{w^2 - 4\mu}\right)^3} dx \tag{3.3.32}$$

Substituting the algebra (3.3.9) with $\alpha_q = \delta_{q,N}$, performing a partial integration on the part containing Q_{ij} and noting that $\frac{\partial^2 w}{\partial u_k \partial u_N} = 0$ finishes the proof of the lemma. This last fact follows from (3.1.27), which ensures that

$$\frac{\partial w}{\partial u_N} = -\frac{P_{u_N}}{P_w} = -1 \tag{3.3.33}$$

$\square$

Denoting the basis of B by $\{\chi_i\}$ we can reformulate the Picard-Fuchs equations as

$$\frac{\partial}{\partial u_i}\chi_j + \sum_k \Gamma^k_{ij}\chi_k = 0 \tag{3.3.34}$$

thus again defining a flat connection. Since

$$\frac{\partial^2 \lambda_{SW}}{\partial u_k \partial u_N} = -\frac{w\frac{\partial w}{\partial u_k}\frac{\partial w}{\partial u_N}}{\left(\sqrt{w^2-4\mu}\right)^3}dx \tag{3.3.35}$$

we can split up the connection $\Gamma^k_{ij} = \Gamma^{(1)k}_{ij} + \Gamma^{(3)k}_{ij}$ according to the number of powers of the square roots occurring in the denominator. For the term with three powers, the flatness condition reduces to the two identities (3.3.29) on the structure constants of the algebra.

It turns out that the flat coordinates t_i from singularity theory precisely cause $\Gamma^{(1)k}_{ij}(t) = 0$, and therefore again get the interpretation of flat coordinates. In terms of them, the Picard-Fuchs equations read

$$\left(\frac{\partial^2}{\partial t_i \partial t_j} - \sum_k C^k_{ij}(t)\frac{\partial^2}{\partial t_k \partial t_N}\right)\oint_\Gamma \lambda_{SW} = 0 \tag{3.3.36}$$

where we integrated over an arbitrary cycle Γ. We can now prove the following theorem

Theorem 3.13. *[32] For simply laced Lie algebras, the prepotential $\mathcal{F}$ and structure constants $C^k_{ij}(\beta, a)$ are related by*

$$\frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} = \sum_{l,m=1}^{N} C^l_{ij}(\beta, a)\beta_m \mathcal{F}_{klm} \tag{3.3.37}$$

Therefore the prepotential $\mathcal{F}(a_1, ..., a_N)$ satisfies the WDVV system.

Proof. Changing the coordinates from t_i to a_i in equation (3.3.36), we find

$$\begin{aligned}\sum_{i,j}\left(\frac{\partial a_i}{\partial t_r}\frac{\partial a_j}{\partial t_s} - \sum_t^N C^t_{rs}(t)\frac{\partial a_i}{\partial t_t}\frac{\partial a_j}{\partial t_N}\right)\frac{\partial^2}{\partial a_i \partial a_j}\oint_\Gamma \lambda_{SW} + \\ \sum_i\left(\frac{\partial^2 a_i}{\partial t_r \partial t_s} - \sum_t C^t_{rs}(t)\frac{\partial^2 a_i}{\partial t_t \partial t_N}\right)\oint_\Gamma \frac{\partial \lambda_{SW}}{\partial a_i} = 0\end{aligned} \tag{3.3.38}$$

Ordinarily, the two halves of this equation need not vanish separately. However, since

$$a_i = \oint_{A_i}\lambda_{SW} \tag{3.3.39}$$

we find that a_i satisfies (3.3.36) and therefore the second half of equation (3.3.38) vanishes. Taking the cycle $\Gamma = B_k$ and defining $\beta_m = \frac{\partial a_m}{\partial t_N}$, the first half can be rewritten as

$$\begin{aligned}\frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} &= \sum_{l,m,r,s,t}\left(\frac{\partial t_r}{\partial a_i}\frac{\partial t_s}{\partial a_j}C^t_{rs}(t)\frac{\partial a_l}{\partial t_t}\right)\left(\frac{\partial a_m}{\partial t_N}\right)\frac{\partial^3 \mathcal{F}}{\partial a_k \partial a_l \partial a_m} \\ &= \sum_{l,m} C^k_{ij}(\beta, a)\beta_m \frac{\partial^3 \mathcal{F}}{\partial a_k \partial a_l \partial a_m}\end{aligned} \tag{3.3.40}$$

□

3.3.4 Picard-Fuchs equations for the non simply laced algebras

For simply laced Lie algebras, the family of associative algebras $\mathcal{A}$ is precisely the Landau-Ginzburg algebra. This gives us the direct connection between the flat coordinates and the algebra, expressed in equation (3.3.36). For non simply laced Lie algebras, the associative algebras are not the Landau-Ginzburg algebras [64]. For example, there is only one Landau-Ginzburg algebra of type BC whereas there are two separate algebras in the Seiberg-Witten context. Nevertheless we can show that for the classical B and C algebras, a similar relation to (3.3.37) still holds, now connecting the Landau-Ginzburg flat coordinates to the Seiberg-Witten algebras. This allows us to continue the proof.

Proposition 3.14. *[27] For the non simply laced Lie algebras of type B_N and C_N the relation (3.3.37) holds. Therefore the corresponding prepotentials satisfy the WDVV equations.*

Proof. We first define the BC Landau-Ginzburg algebra. In terms of its flat coordinates the multiplication structure reads

$$\begin{aligned} \phi_i(t) &= -\frac{\partial W^{BC}}{\partial t_i} \\ \phi_i(t)\phi_j(t) &= \hat{C}_{ij}^k(t)\phi_k(t) + Q_{ij}W_x^{BC} \end{aligned} \tag{3.3.41}$$

Furtermore, it is not hard to show that Q_{ij} is divisable by x and we express Q_{ij} as a linear combination

$$Q_{ij} = x\sum_k \hat{D}_{ij}^k(t)\phi_k \tag{3.3.42}$$

In [32] the following set of Picard-Fuchs equations was obtained

$$\begin{aligned} \Bigg(\frac{\partial^2}{\partial t_i \partial t_j} - \sum_{k=1}^N \hat{C}_{ij}^k(t)\frac{\partial^2}{\partial t_k \partial t_N} - \sum_{k=1}^N\sum_{n=1}^N \frac{\epsilon d_n t_n}{h_{\mathfrak{g}}^\vee}\hat{D}_{ij}^k\frac{\partial^2}{\partial t_k\partial t_n} + \\ \sum_{k=1}^N \hat{D}_{ij}^k \frac{1}{h_{\mathfrak{g}}^\vee}\left(1-d_k\right)\frac{\partial}{\partial t_k}\Bigg)\oint_\Gamma \lambda_{SW} = 0 \end{aligned} \tag{3.3.43}$$

where the $\hat{C}_{ij}^k(t)$ are the structure constants of the BC Landau-Ginzburg theory, the d_n are the degrees of the Lie algebra and $\epsilon = 1(-1)$ for $B_N(C_N)$. Making a change of coordinates to the a_i just like we did for simply laced algebras and using the fact that the a_i satisfy (3.3.43), we get

$$\sum_{i,j}\left[\frac{\partial a_i}{\partial t_r}\frac{\partial a_j}{\partial t_s} - \sum_t \hat{C}_{rs}^t(t)\frac{\partial a_i}{\partial t_t}\frac{\partial a_j}{\partial t_N} - \sum_{k,n}\hat{D}_{rs}^t\frac{\epsilon d_n t_n}{h_{\mathfrak{g}}^\vee}\frac{\partial a_i}{\partial t_n}\frac{\partial a_j}{\partial t_t}\right]\frac{\partial^2}{\partial a_i \partial a_j}\oint_\Gamma \lambda_{SW} = 0 \tag{3.3.44}$$

Unfortunately, this is not in the form of (3.3.38) and we cannot continue as before. We do see however that the fourth term in (3.3.43) does not contribute to (3.3.44). So we go back to the first three terms of (3.3.43) and with the benefit of hindsight we introduce new objects $\gamma_{ij}^k(t)$ as

$$\hat{C}_{ij}^k(t) = \gamma_{ij}^k(t) - \sum_{k,q}\hat{D}_{ij}^l\frac{\epsilon d_n t_n}{h_{\mathfrak{g}}^\vee}\gamma_{nl}^k \tag{3.3.45}$$

We substitute this into the first three terms of (3.3.43) and obtain

$$\left(\frac{\partial^2}{\partial t_i \partial t_j} - \gamma_{ij}^k(t)\frac{\partial^2}{\partial t_k \partial t_N}\right)\oint_\Gamma \lambda_{SW} + \\ \sum_{l,n} \hat{D}_{ij}^l \frac{\epsilon d_n t_n}{h_{\mathfrak{g}}^\vee}\left(\frac{\partial^2}{\partial t_l \partial t_n} - \gamma_{ln}^k(t)\frac{\partial^2}{\partial t_k \partial t_N}\right)\oint_\Gamma \lambda_{SW} \quad (3.3.46)$$

This expression consists of two parts. Making the change of coordinates to the a variables gives two equations that have to vanish separately, one for each of the two parts of (3.3.46). Each of these equations then boils down to the relation

$$\mathcal{F}_{ijk} = \gamma_{ij}^l(a)\frac{\partial a_m}{\partial t_N}\mathcal{F}_{klm} \quad (3.3.47)$$

and proves that the WDVV equations hold if the $\gamma_{ij}^k(t)$ are well-defined and if they are the structure constants of some associative algebra. This is the subject of the following lemma.

Lemma 3.15. *The objects $\gamma_{ij}^k(t)$ defined through relation (3.3.45) exist and they are precisely the structure constants $C_{ij}^k(t)$ of the Seiberg-Witten algebra in terms of the coordinates t_i. The Seiberg-Witten algebras were defined separately for B_N and C_N in section 3.3.2.*

Proof. We will restrict ourselves to the B_N case here, the proof for C_N is very similar. We will rewrite (3.3.41) in such a way that it becomes of the form

$$\phi_i(t)\phi_j(t) = \sum_{k=1}^r \gamma_{ij}^k(t)\phi_k(t) + R_{ij}\left[x\partial_x W^{BC} - W^{BC}\right] \quad (3.3.48)$$

As a first step, we use (3.3.45):

$$\begin{aligned} \phi_i\phi_j &= \left[\hat{C}_i \cdot \overrightarrow{\phi} + \hat{D}_i \cdot \overrightarrow{\phi} x\partial_x W_{BC}\right]_j \\ &= \left[\left(\gamma_i - \hat{D}_i \cdot \sum_{n=1}^r \frac{2nt^n}{2r-1}\gamma_n\right)\cdot \overrightarrow{\phi} + \hat{D}_i \cdot \overrightarrow{\phi} x\partial_x W_{BC}\right]_j \\ &= \left[\gamma_i \cdot \overrightarrow{\phi} - \hat{D}_i \cdot \sum_{n=1}^r \frac{2nt^n}{2r-1}\gamma_n \cdot \overrightarrow{\phi} + \hat{D}_i \cdot \overrightarrow{\phi} xW^{BC}\right]_j \end{aligned} \quad (3.3.49)$$

The notation $\overrightarrow{\phi}$ stands for the vector with components ϕ_k and we use a matrix notation for the structure constants. There are two things about this equation that we would like to change: the first thing is that we want the structure constants to be defined by the first term, so we would like the middle term to vanish. The second thing is that we want the third term to contain the generator $W^{BC} - xW_x^{BC}$ of the ideal J. As a first step towards resolving both these problems, we will take part of the third term and cancel it with the middle term. To do this, we will use the following equation which expresses that W^{BC} is homogeneous in the Lie algebraic grading

$$xW_x^{BC} + \sum_n 2nt_n \frac{\partial W^{BC}}{\partial t_n} = 2NW^{BC} \quad (3.3.50)$$

Using this equation we can cancel the middle term of (3.3.49) with part of the third term at the expense of introducing new terms which then have to be canceled etcetera. This recursive process will end however and yield the desired result. First we split up the third term of (3.3.49) as follows

$$\begin{aligned}\left[\hat{D}_i \cdot \overrightarrow{\phi}\, xW_x^{BC}\right]_j &= \left[-\frac{1}{2N-1}\hat{D}_i \cdot \overrightarrow{\phi}\, xW_x^{BC} + \left(1+\frac{1}{2N-1}\right)\hat{D}_i \cdot \overrightarrow{\phi}\, xW_x^{BC}\right]_j \\ &= \left[-\frac{\hat{D}_i}{2N-1}\cdot\overrightarrow{\phi}\left(2NW^{BC} - \sum_{n=1}^{N} 2nt_n\phi_n\right) + \frac{2N\hat{D}_i}{2N-1}\cdot\overrightarrow{\phi}\, xW_x^{BC}\right]_j \end{aligned} \tag{3.3.51}$$

Using the Landau-Ginzburg algebra (3.3.41) we rewrite the products of ϕ occurring here, thus rewriting (3.3.49) as

$$\begin{aligned}\phi_i\phi_j = \Bigg[&\gamma_i \cdot \overrightarrow{\phi} - \frac{\hat{D}_i}{2N-1}\cdot\sum_n 2nt_n\left(\gamma_n\cdot\overrightarrow{\phi} - \hat{C}_n\cdot\overrightarrow{\phi}\right) - \\ &\frac{\hat{D}_i}{2N-1}\cdot\sum_n 2nt_n\hat{D}_n\cdot\overrightarrow{\phi}\, xW_x^{BC}\Bigg]_j + \frac{2N\hat{D}_i}{2N-1}\cdot\left[xW_x^{BC} - W_{BC}\right]_j\end{aligned} \tag{3.3.52}$$

We now use (3.3.45) again to rewrite the second term in the first line. Then we find

$$\begin{aligned}\phi_i\phi_j = \Bigg[&\gamma_i \cdot \overrightarrow{\phi} - \frac{\hat{D}_i}{2N-1}\cdot\sum_n 2nt_n\Bigg(-\hat{D}_n\cdot\sum_m \frac{2mt^m}{2N-1}\gamma_m\cdot\overrightarrow{\phi} + \\ &\hat{D}_n\cdot\overrightarrow{\phi}\, xW_x^{BC}\Bigg)\Bigg]_j + \frac{2N\hat{D}_i}{2N-1}\cdot\left[xW_x^{BC} - W_{BC}\right]_j\end{aligned} \tag{3.3.53}$$

Note that by cancelling one term, we automatically calculate modulo $xW_x^{BC} - W_{BC}$. We can now repeat the whole process on the term between round brackets in the first line of (3.3.53). This is a recursive process and each step will introduce an extra factor of $\hat{D}_i$. To see that the recursive process stops, we will prove that the $\hat{D}_i$ are nilpotent matrices.

The degree of Q_{ij} is $[Q_{ij}] = 2N+1-2(i+j)$. Dividing by x the degree becomes $2N - 2(i+j)$. Since $[\phi_k] = 2N-2k$ one cannot divide $\frac{Q_{ij}}{x}$ by ϕ_k for $j \geq k$ and therefore the matrix $\hat{D}_i$ defined in (3.3.42) is lower triangular and thus nilpotent. □

□

3.3.5 Duality transformations and Picard-Fuchs equations

The proof of the WDVV equations by means of the Picard-Fuchs equations makes particularly clear the role that the duality transformations, discussed in section 1.2.4, play in Seiberg-Witten theory [25]. We have defined the special $2N$ cycles on the family of Riemann surface as part of a canonical basis of cycles, and the proof of the WDVV equations does not change at all if we apply a symplectic transformation to the cycles. So although different choices of cycles give different prepotentials, all of these prepotentials will satisfy the WDVV system (1.2.2).

3.3.6 The residue formula

An alternative to the approach of Picard-Fuchs equations is given by the residue formula [47], whose origins lie in the theory of integrable systems [39].

A common way of proving Riemann's bilinear relations on a Riemann surface Σ is to cut open the surface to obtain a fundamental $4g$-sided polygon Π and use Cauchy's residue theorem on Π. We will use the same method to obtain a residue formula for the third order derivatives of $\mathcal{F}$.

We start by rewriting $\mathcal{F}_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k}$ as follows

$$\begin{aligned}
\mathcal{F}_{ijk} = \frac{\partial}{\partial a_k}\mathcal{F}_{ij} = \frac{\partial}{\partial a_k}\sum_m \oint_{A_m}\omega_i \oint_{B_m}\omega_j = \\
\sum_m \oint_{A_m}\frac{\partial \omega_i}{\partial a_k}\oint_{B_m}\omega_j + \sum_m \oint_{A_m}\omega_i \oint_{B_m}\frac{\partial \omega_j}{\partial a_k} = 0 + \sum_m \oint_{A_m}\omega_i \oint_{B_m}\frac{\partial \omega_j}{\partial a_k} = \\
\sum_m \left(\sum_m \oint_{A_m}\omega_i \oint_{B_m}\frac{\partial \omega_j}{\partial a_k} - \sum_m \oint_{B_m}\omega_i \oint_{A_m}\frac{\partial \omega_j}{\partial a_k} \right) = \sum \text{res}\left(\chi_i \frac{\partial \omega_j}{\partial a_k} \right)
\end{aligned} \tag{3.3.54}$$

where χ_i is a function which is single valued on Π but not on Σ, defined in such a way that $d\chi_i = \omega_i$ on Π. It is always possible to find such χ_i since Π is simply connected, and therefore the holomorphic differential $\omega_i = \frac{\partial \lambda_{SW}}{\partial a_i}$ is exact on Π.

In the derivation of (3.3.54) essential use has been made of the fact that

$$\oint_{A_i} \frac{\partial \lambda_{SW}}{\partial a_j} = \delta_{ij} \tag{3.3.55}$$

This relation holds for all Lie algebras due to the particular construction of cycles in section 3.1.5 and lemma 3.9. We can work out $\mathcal{F}_{ijk}$ further and find

Proposition 3.16. *[47] The following residue formula holds*

$$\mathcal{F}_{ijk} = \sum \textit{res}\left(\frac{\omega_i \otimes \omega_j \otimes \omega_k}{dx \otimes \frac{dz}{z}} \right) = \sum \textit{res}\left(\frac{P_{a_i} P_{a_j} P_{a_k}}{(zP_z)^2 P_x} \right) \tag{3.3.56}$$

Proof. We can calculate $\frac{\partial \omega_j}{\partial a_k} = \frac{\partial^2 \lambda_{SW}}{\partial a_j \partial a_k}$ keeping in mind that we can throw away any terms that do not contribute to the residue. Due to the second differentiation of λ_{SW}, poles arise at the zeroes of P_x. These mark the branch points of the curve, so we need precisely two factors P_x in the denominator to get a contribution to the residue. We then find up to terms that do not contribute to the residue

$$\begin{aligned}
\frac{\partial^2 \lambda_{SW}}{\partial a_j \partial a_k} &= -\frac{\partial^2 x}{\partial a_j \partial a_k}\frac{dz}{z} = \frac{\partial}{\partial a_j}\left(\frac{P_{a_k}}{P_x}\right)\frac{dz}{z} \\
&= \frac{P_{a_j a_k}}{P_x}\frac{dz}{z} - \frac{d}{dx}\left(\frac{P_{a_j}P_{a_k}}{P_x}\right)\frac{dz}{zP_x} \\
&= -\frac{d}{dx}\left(\frac{P_{a_j}P_{a_k}}{P_x}\right)\frac{dz}{zP_x}
\end{aligned} \tag{3.3.57}$$

Performing a partial integration we find [47]

$$\sum \text{res}\left(\chi_i \frac{\partial \omega_j}{\partial a_k}\right) = \sum \text{res}\left(-\chi_i \frac{d}{dx}\left(\frac{P_{a_j} P_{a_k}}{P_x}\right)\frac{dz}{zP_x}\right) =$$
$$\sum \text{res}\left(\frac{d\chi_i}{dx}\frac{P_{a_j}P_{a_k}}{P_x^2}\frac{dz}{z}\right) = \sum \text{res}\left(\frac{P_{a_i}P_{a_j}P_{a_k}}{(zP_z)^2 P_x} dx\right) \quad (3.3.58)$$

and this ends the proof. □

In the proof of the residue formula, the calculation of the second order derivatives of λ_{SW} is similar to the one for the Picard-Fuchs method. The crucial difference however is that some terms can be neglected because they do not contribute to the residue. This makes the residue formula applicable also for the non simply laced Lie algebras. After having obtained the above proposition, the proof that $\mathcal{F}$ satisfies the WDVV system becomes trivial.

Corollary 3.17. *The relation (3.3.37) follows from the definition of the algebra together with the residue formula of proposition 3.16. Therefore we conclude again that the prepotential $\mathcal{F}$ satisfies the WDVV system, using now the residue formula instead of the Picard-Fuchs equations.*

3.3.7 Representation independence of the family of associative algebras

We have shown in section 3.1.5 that the period integrals of λ_{SW} over the first N cycles of type A and the first N cycles of type B are independent of the representation ρ of $\mathfrak{g}$ chosen to define the family of spectral curves. Therefore also the prepotential $\mathcal{F}$ and the proof of the WDVV equations are representation independent.

Since a family of associative algebras is connected to a function satisfying the WDVV equations, this strongly suggests that the family $\mathcal{A}$ defined in section 3.3.2 exists for any representation and is independent of it. If so, then the spectral equation

$$P(x, w, u_i) = 0 \quad (3.3.59)$$

implicitly defines a one-variable Landau-Ginzburg superpotential $w(x, u_i)$.

Proposition 3.18. *For any irreducible representation ρ the family $\mathcal{A}$ of algebras, see (3.3.10), is defined and is independent of ρ. Therefore the implicitly defined function $w(x, u_i)$ is a one-variable Landau-Ginzburg superpotential for any ρ.*

Proof. Since the period integrals of λ_{SW} are representation independent, the derivation of the residue formula (3.3.56) is representation independent. Since the WDVV equations hold, we find that

$$\sum \text{res}\left(\frac{\omega_i \otimes \omega_j \otimes \omega_k}{dx \otimes \frac{dz}{z}}\right) = \sum_{k,l} C_{ij}^k(a)\frac{\partial a_l}{\partial t_N}\sum \text{res}\left(\frac{\omega_k \otimes \omega_l \otimes \omega_m}{dx \otimes \frac{dz}{z}}\right) \quad (3.3.60)$$

thus showing that the algebra (3.3.10)

$$\omega_i \otimes \omega_j = \sum_{k,l} C_{ij}^k(a)\frac{\partial a_l}{\partial t_N}\omega_k \otimes \omega_l \qquad \text{mod } \frac{dz}{z} \quad (3.3.61)$$

is representation independent. □

As an example, we will consider the Lie algebra A_4 in the 5, 10 and 24 dimensional representations[5]. The spectral curves are given by

$$P_5 \quad = \quad w + x^5 + u_1 x^3 + u_2 x^2 + u_3 x + u_4 \tag{3.3.62}$$

$$\begin{aligned} P_{10} \quad = \quad & w^2 + \left(-11x^5 - 4u_1x^3 - 7u_2x^2 + (-u_1^2 + 4u_3)x + 2u_4 - u_1u_2\right)w \\ & -x^{10} - 3x^8u_1 + x^7u_2 + (-3u_1^2 + 3u_3)x^6 + (-11u_4 + 2u_1u_2)x^5 + \\ & (u_2^2 + 2u_1u_3 - u_1^3)x^4 + (-4u_2u_3 - 4u_4u_1 + u_1^2u_2)x^3 + \\ & (-7u_4u_2 + u_2^2u_1 - u_1^2u_3 + 4u_3^2)x^2 + (-u_2^3 + 4u_4u_3 - u_4u_1^2)x - \\ & u_4^2 + u_2^2u_3 - u_4u_1u_2 \end{aligned} \tag{3.3.63}$$

5 The 24-dimensional adjoint representation is not miniscule and we consider only the part of the curve containing the highest weight

$$\begin{aligned}
P_{24} \quad = \quad & \mathbf{x}^{20} + 10u_1\mathbf{x}^{18} + (39u_1^2 + 10u_3)\mathbf{x}^{16} + \\
& (105u_1^3 + 25u_2^2 + 25(-u_1^2 + 2u_3)u_1)\mathbf{x}^{14} + \\
& (533/4u_1^4 + 92u_1u_2^2 + 200(u_4 + \mathbf{w})u_2 + \\
& \qquad 29/2(-u_1^2 + 2u_3)u_1^2 - 95/4(-u_1^2 + 2u_3)^2)\mathbf{x}^{12} + \\
& (74u_1^5 + 248u_1^2u_2^2 + 400(u_4 + \mathbf{w})u_1u_2 - 82(-u_1^2 + 2u_3)u_1^3 + \\
& \qquad 625(u_4 + \mathbf{w})^2 + 130(-u_1^2 + 2u_3)u_2^2 - 90(-u_1^2 + 2u_3)^2u_1)\mathbf{x}^{10} + \\
& (-7/2u_1^6 + 406u_1^3u_2^2 - 235(u_4 + \mathbf{w})u_1^2u_2 - 149(-u_1^2 + 2u_3)u_1^4 - \\
& \qquad 53u_2^4 + 354(-u_1^2 + 2u_3)u_1u_2^2 - 475(-u_1^2 + 2u_3)(u_4 + \mathbf{w})u_2 - \\
& \qquad 231/2(-u_1^2 + 2u_3)^2u_1^2 + 1750(u_4 + \mathbf{w})^2u_1 + 5(-u_1^2 + 2u_3)^3)\mathbf{x}^8 + \\
& (-30u_1^7 + 883/2u_1^4u_2^2 - 995(u_4 + \mathbf{w})u_1^3u_2 - 131(-u_1^2 + 2u_3)u_1^5 - \\
& \qquad 102u_1u_2^4 + 625(u_4 + \mathbf{w})^2u_1^2 + 700(u_4 + \mathbf{w})u_2^3 + \\
& \qquad 591(-u_1^2 + 2u_3)u_1^2u_2^2 - 1075(-u_1^2 + 2u_3)(u_4 + \mathbf{w})u_1u_2 - \\
& \qquad 107(-u_1^2 + 2u_3)^2u_1^3 - 1875(-u_1^2 + 2u_3)(u_4 + \mathbf{w})^2 + \\
& \qquad 285/2(-u_1^2 + 2u_3)^2u_2^2 - 10(-u_1^2 + 2u_3)^3u_1)\mathbf{x}^6 + \\
& (-47/4u_1^8 + 302u_1^5u_2^2 - 535(u_4 + \mathbf{w})u_1^4u_2 - 61/2(-u_1^2 + 2u_3)u_1^6 + \\
& \quad 55(-u_1^2 + 2u_3)^3u_1^2 - 750(u_4 + \mathbf{w})^2u_1^3 + 600(u_4 + \mathbf{w})u_1u_2^3 + \\
& \quad 550(-u_1^2 + 2u_3)u_1^3u_2^2 + 1875(u_4 + \mathbf{w})^2u_2^2 + 45/4(-u_1^2 + 2u_3)^2u_1^4 - \\
& \quad 325(-u_1^2 + 2u_3)(u_4 + \mathbf{w})u_1^2u_2 - 135(-u_1^2 + 2u_3)u_2^4 - \\
& \quad 2500(-u_1^2 + 2u_3)(u_4 + \mathbf{w})^2u_1 + 240(-u_1^2 + 2u_3)^2u_1u_2^2 + \\
& \quad 250(-u_1^2 + 2u_3)^2(u_4 + \mathbf{w})u_2 - 180u_1^2u_2^4 + 25(-u_1^2 + 2u_3)^4)\mathbf{x}^4 + \\
& (u_1^9 + 183/2u_1^6u_2^2 - 115(u_4 + \mathbf{w})u_1^5u_2 + 14(-u_1^2 + 2u_3)u_1^7 - 95u_1^3u_2^4 + \\
& \quad 50(u_4 + \mathbf{w})^2u_1^4 - 165(u_4 + \mathbf{w})u_1^2u_2^3 + 190(-u_1^2 + 2u_3)u_1^4u_2^2 + \\
& \quad 27u_2^6 + 3000(u_4 + \mathbf{w})^2u_1u_2^2 - 65(-u_1^2 + 2u_3)(u_4 + \mathbf{w})u_1^3u_2 + \\
& \quad 45(-u_1^2 + 2u_3)^2u_1^5 - 99(-u_1^2 + 2u_3)u_1u_2^4 - 6250(u_4 + \mathbf{w})^3u_2 + \\
& \quad 625(-u_1^2 + 2u_3)(u_4 + \mathbf{w})^2u_1^2 - 225(-u_1^2 + 2u_3)(u_4 + \mathbf{w})u_2^3 + \\
& \quad 187/2(-u_1^2 + 2u_3)^2u_1^2u_2^2 + 50(-u_1^2 + 2u_3)^2(u_4 + \mathbf{w})u_1u_2 + \\
& \quad 52(-u_1^2 + 2u_3)^3u_1^3 + 1250(-u_1^2 + 2u_3)^2(u_4 + \mathbf{w})^2 - \\
& \quad 5(-u_1^2 + 2u_3)^3u_2^2 + 20(-u_1^2 + 2u_3)^4u_1)\mathbf{x}^2 - \\
& 96(u_4 + \mathbf{w})u_1^6u_2 - 27/4u_1^4u_2^4 + 158(u_4 + \mathbf{w})^2u_1^5 + 2u_1^{10} + \\
& 52(-u_1^2 + 2u_3)u_1^5u_2^2 + 1950(u_4 + \mathbf{w})^2u_1^2u_2^2 + 38(-u_1^2 + 2u_3)^2u_1^6 - \\
& 356(-u_1^2 + 2u_3)(u_4 + \mathbf{w})u_1^4u_2 + 500(-u_1^2 + 2u_3)^2(u_4 + \mathbf{w})^2u_1 - \\
& 27/2(-u_1^2 + 2u_3)u_1^2u_2^4 + 17u_1^7u_2^2 - 3750(u_4 + \mathbf{w})^3u_1u_2 + \\
& 14(-u_1^2 + 2u_3)u_1^8 - 315(-u_1^2 + 2u_3)(u_4 + \mathbf{w})u_1u_2^3 + 3125(u_4 + \mathbf{w})^4 + \\
& 53(-u_1^2 + 2u_3)^2u_1^3u_2^2 + 1125(-u_1^2 + 2u_3)(u_4 + \mathbf{w})^2u_2^2 - \\
& 460(-u_1^2 + 2u_3)^2(u_4 + \mathbf{w})u_1^2u_2 + 50(-u_1^2 + 2u_3)^3u_1^4 - \\
& 27/4(-u_1^2 + 2u_3)^2u_2^4 + 108(u_4 + \mathbf{w})u_2^5 - 299(u_4 + \mathbf{w})u_1^3u_2^3 + \\
& 18(-u_1^2 + 2u_3)^3u_1u_2^2 - 200(-u_1^2 + 2u_3)^3(u_4 + \mathbf{w})u_2 + \\
& 32(-u_1^2 + 2u_3)^4u_1^2 + 550(-u_1^2 + 2u_3)(u_4 + \mathbf{w})^2u_1^3 + 8(-u_1^2 + 2u_3)^5
\end{aligned} \tag{3.3.64}$$

Defining the ideal $I = \langle P, P_x \rangle \subset \mathbf{C}[x,w]$, explicit computations show that indeed the subalgebras of $\mathbf{C}[x,w]/I$ generated by the P_{u_i} have *precisely* the same structure constants (3.3.14).

3.4 The perturbative limit for type A Lie algebras

We have claimed in section 2.1 that the nonperturbative prepotential $\mathcal{F}$ implicitly defined through

$$a_k = \oint_{A_k} \lambda_{SW} \qquad \frac{\partial \mathcal{F}}{\partial a_j} = \oint_{B_j} \lambda_{SW} \tag{3.4.1}$$

can be expanded in terms of the parameter μ as

$$\mathcal{F} = \mathcal{F}_{class} + \mathcal{F}_{pert} + \sum_{k=1}^{\infty} c_k(a)\mu^k \tag{3.4.2}$$

Here $\mathcal{F}_{class}$ is a quadratic polynomial in the a_i which is irrelevant for the WDVV equations, and $\mathcal{F}_{pert}$ is the perturbative limit that was studied extensively in chapter 2. Following [15] we will show explicitly in the case of Lie algebra A_N that the nonperturbative prepotential is of the form (3.4.2) and the perturbative part is identified as

$$\mathcal{F}_{pert} = \frac{-1}{4} \sum_{i,j=1}^{N+1} (a_i - a_j)^2 \log \left(\frac{(a_i - a_j)^2}{\mu}^2 \right) \tag{3.4.3}$$

where $\sum_i a_i = 0$. This is the expression appearing in (2.2.3).

We start by fixing the choice of cycles for the hyperelliptic A_N curve (3.1.4). The branch points of the curve, viewed as defining $y(x)$, are given by

$$W(x) \pm 2\mu = \prod_{i=1}^{N+1} (x - e_i) \pm 2\mu = 0 \tag{3.4.4}$$

where $\sum_i e_i = 0$. Denoting the branch points by $x_k^{\pm}$, we let the cycle A_k run on the first sheet around the branch cut from x_k^- to x_k^+. This gives us the cycles $A_1, ..., A_N$. We define B_k by the cycle running from x_{N+1}^- to x_k^- on the first sheet and back from x_k^- to x_{N+1}^- on the second sheet. The Seiberg-Witten differential, whose period integrals around these cycles we will calculate, is given by (3.1.8)

$$\lambda_{SW} = \frac{xW' dx}{W\sqrt{1 - \frac{4\mu^2}{W^2}}} \tag{3.4.5}$$

where we forget the term $\log(2)dx$ because it doesn't contribute significantly to the period integrals. In the limit $\mu \to 0$, the branch cut from x_k^- to x_k^+ shrinks to the single point e_k. The cycle A_k is not affected by taking this limit, since the homology class determines the outcome of the period integrals and therefore we can keep the cycle fixed. So we can make an expansion of the period integral a_k in powers of μ by expanding λ_{SW}. This is done by a

Taylor series expansion in μ which converges for small μ because the cycle A_k never comes close to the points e_i where W is small. The expansion of λ_{SW} reads

$$\lambda_{SW} = \sum_{m=0}^{\infty} \alpha_m (2\mu)^{2m} \frac{xW'dx}{W^{2m+1}} \tag{3.4.6}$$

where

$$\alpha_m = \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m+1)} \tag{3.4.7}$$

This reduces λ_{SW} to a rational differential on a sphere with punctures at the e_k and we can evaluate the period integral with the Cauchy residue theorem. The expansion of a_k is therefore

$$\begin{aligned} a_k &= \oint_{A_k} \frac{xW'dx}{W} + \sum_{m\geq 1} \alpha_m (2\mu)^{2m} \oint_{A_k} \frac{xW'dx}{W^{2m+1}} \\ &= \oint_{A_k} \sum_{i=1}^{N+1} \frac{xdx}{x-e_i} + \sum_{m\geq 1} \alpha_m (2\mu)^{2m} \oint_{A_k} \frac{1}{2m} \left(-d \left[\frac{x}{W^{2m}} \right] + \frac{dx}{W^{2m}} \right) \\ &= e_k + \sum_{m\geq 1} \alpha_m (2\mu)^{2m} \oint_{A_k} \frac{dx}{W^{2m}} \end{aligned} \tag{3.4.8}$$

Here the residue of the first term was calculated in e_k. The residue of the second term can be found in closed form by noting that

$$\begin{aligned} \frac{1}{W^{2m}} = \frac{1}{(x-e_k)^{2m}} \prod_{j\neq k} \frac{1}{(x-e_j)^{2m}} = \frac{1}{(x-e_k)^{2m}} S_k(x)^m = \\ \sum_{p=-2m}^{\infty} \frac{1}{(2m+p)!} \frac{\partial^{2m+p} S_k^m (x=e_k)}{\partial x^{2m+p}} (x-e_k)^p \end{aligned} \tag{3.4.9}$$

where we used a Taylor series expansion of $S_k^m(x) = \prod_{j\neq k} \frac{1}{(x-e_j)^{2m}}$ around $x = e_k$. We find the residue to be

$$a_k = e_k + \sum_{m\geq 1} \frac{\alpha_m}{2m(2m-1)!} (2\mu)^{2m} \frac{\partial^{2m-1} S_k(e_k)^m}{\partial e_k^{2m-1}} \tag{3.4.10}$$

The evaluation of the period integrals over the B cycles is more delicate. Since these cycles run from x^-_{N+1} to x^-_k on one sheet and back again on the other, we cannot take the limit $\mu \to 0$ and at the same time avoid the points e_k where $W = 0$. The expansion (3.4.6) of λ_{SW} is therefore not valid because the expansion parameter $\frac{2\mu}{W}$ can become big. To work around this problem, we introduce an auxiliary parameter ξ and consider the ξ-dependent integral

$$b_k(\xi) = \oint_{B_k} \frac{xW'dx}{W\sqrt{1-\left(\frac{2\xi\mu}{W}\right)^2}} \tag{3.4.11}$$

The idea is to use the fact that the differential has a convergent power series expansion in ξ for small values of $|\xi|$ and normal values of μ, do some manipulations on the integral and then make an analytic continuation to $\xi = 1$ in the end. This should give a valid series expansion of b_k in μ.

The integral of λ_{SW} on the second sheet equals the integral on the first sheet because on the second sheet $\sqrt{W^2 - 4\mu^2} = -W\sqrt{1 - \frac{4\mu}{W^2}}$ (which gives a minus sign) and the orientation of the integration curve is reversed (which gives another minus sign). Therefore

$$
\begin{aligned}
b_k(\xi) &= 2\int_{x_{N+1}^-}^{x_k^-} \frac{xW'dx}{W\sqrt{1-\left(\frac{2\xi\mu}{W}\right)^2}} \\
&= 2\int_{x_{N+1}^-}^{x_k^-} \sum_{m=0}^{\infty} \alpha_m \xi^{2m}(2\mu)^{2m} \frac{xW'dx}{W^{2m+1}} = 2\int_{x_{N+1}^-}^{x_k^-} \sum_{i=1}^{N+1} \frac{xdx}{x-e_i} + \\
&\qquad 2\sum_{m\geq 1} \alpha_m \xi^{2m}(2\mu)^{2m} \int_{x_{N+1}^-}^{x_k^-} \frac{1}{2m}\left(-d\left[\frac{x}{W^{2m}}\right] + \frac{dx}{W^{2m}}\right)
\end{aligned}
\tag{3.4.12}
$$

Whenever we calculate parts of this integral explicitly, we will only keep the terms from the upper boundary x_k^-, remembering that there is a similar contribution from x_{N+1}^- accompanying it. Since the branch point equation (3.4.4) gives $W(x_k^-) = 2\mu$, we find that

$$
\begin{aligned}
b_k(\xi) &= 2(N+1)x_k^- + 2\sum_{j=1}^{N+1} e_j \log(x_k^- - e_j) - \\
&\qquad 2\sum_{m\geq 1} \frac{\alpha_m}{2m}\xi^{2m} x_k^- + 2\sum_{m\geq 1} \frac{\alpha_m}{2m}\xi^{2m}(2\mu)^{2m} \int_{x_{N+1}^-}^{x_k^-} \frac{dx}{W^{2m}}
\end{aligned}
\tag{3.4.13}
$$

The third term contains a series in ξ which converges for $\xi = 1$, so the analytic continuation for that term is trivial. To evaluate the last term, we use an expansion of $\frac{1}{W^{2m}}$ in terms of partial fractions

$$
\frac{1}{W^{2m}} = \sum_{l=1}^{N+1} \sum_{p=-2m}^{0} Q_{l,p}^{(2m)} (x - e_l)^p \tag{3.4.14}
$$

Using again the Taylor expansion (3.4.9) we recognize the coefficients $Q_{l,p}^{(2m)}$ as

$$
Q_{l,p}^{(2m)} = \frac{1}{(2m+p)!} \frac{\partial^{2m+p} S_l(e_l)^m}{\partial e_l^{2m+p}} \tag{3.4.15}
$$

This expansion of $\frac{1}{W^{2m}}$ allows a separation of the last term in (3.4.13) according to the powers $(x - e_l)^p$ occurring in it. For example, the series for $p = -1$ converges for $\xi = 1$

because it equals

$$2\sum_{l=1}^{N+1}\left(\int_{x_{N+1}^-}^{x_k^-}\frac{1}{x-e_l}\right)\sum_{m\geq 1}\frac{\alpha_m}{2m(2m-1)!}(2\mu)^{2m}\frac{\partial^{2m-1}S_l(e_l)^m}{\partial e_l^{2m-1}} = 2\sum_{l=1}^{N+1}(a_l-e_l)\log(x_k^- - e_l) \quad (3.4.16)$$

For each p, one can check [15] that the series converges for $\xi = 1$. Moreover, one can show that the terms for $p \leq -2$ can be expressed by power series in μ. To see this, we note that the terms $(x_k^- - e_k)^{p+1} \sim \mu^{p+1}$ which occur after the integration of $\frac{1}{W^{2m}}$ are singular, but on the other hand there are enough compensating terms μ^{2m} to soak up the singularities.

To show that (3.4.2) holds, the remaining task is to identify the part containing the perturbative prepotential. Forgetting the terms that contribute to the classical and $\mathcal{O}(\mu)$ parts, we focus on the first three terms of (3.4.13) together with the $p=-1$ part of the fourth term

$$Z_k = (2(N+1) - 2\log(2))\, x_k^- + 2\sum_{j=1}^{N+1} a_j \log(x_k^- - e_j) \quad (3.4.17)$$

where we have used the summation formula $\sum_{m\geq 1}\frac{\alpha_m}{2m} = \log(2)$. We remind the reader once again that similar terms with x_k^- replaced by x_{N+1}^- have been omitted. Using (3.4.4) we rewrite Z_k as

$$\begin{aligned} Z_k &= (2(N+1) - 2\log(2))\, x_k^- + 2a_k\log(x_k^- - e_k) + 2\sum_{j\neq k} a_j\log(x_k^- - e_j) = \\ &(2(N+1) - 2\log(2))\, x_k^- + 2a_k\log(2) + 2a_k\log(\mu) - \\ &2\sum_{j\neq k} a_k \log(x_k^- - e_j) + 2\sum_{j\neq k} a_j\log(x_k^- - e_j) = \\ &2(N+1)x_k^- + 2\log(2)(a_k - x_k^-) + 2a_k\log(\mu) - 2\sum_{j\neq k}(a_k - a_j)\log(x_k^- - e_j) \end{aligned} \quad (3.4.18)$$

Using once more (3.4.4) together with (3.4.8) we can express everything in terms of the a_k as follows

$$Z_k = (2(N+1) + 2\log(\mu))\, a_k - 2\sum_{j\neq k}(a_k - a_j)\log(a_k - a_j) + \mathcal{O}(\mu) \quad (3.4.19)$$

Keeping only the terms of the perturbative part and reintroducing the contributions from the lower bound of the integral, we end up with

$$\begin{aligned} &2a_k\log(\mu) - 2\sum_{j\neq k}(a_k - a_j)\log(a_k - a_j) - \\ &2a_{N+1}\log(\mu) + 2\sum_{j\neq N+1}(a_{N+1} - a_j)\log(a_{N+1} - a_j) \end{aligned} \quad (3.4.20)$$

We want to write this as the derivative $\frac{d\mathcal{F}_{pert}}{da_k}$ of the perturbative prepotential (3.4.3). There is a technical complication (typical for the type A_N case) due to $\sum_i e_i = 0$ which implies

$\sum_{i=1}^{N+1} a_i = 0$. Substituting $a_{N+1} = -\sum_{i=1}^{N} a_i$ in $\mathcal{F}_{pert}$ we see that

$$\frac{d\mathcal{F}_{pert}}{da_k} = \frac{\partial\mathcal{F}_{pert}}{\partial a_k} - \frac{\partial\mathcal{F}_{pert}}{\partial a_{N+1}} \tag{3.4.21}$$

Substituting (3.4.3) in this expression indeed gives (3.4.20) up to polynomials in the a_i and up to a redefinition $\mu \to \mu^{h_g^\vee}$ where $h_g^\vee = N+1$ is the dual Coxeter number of A_N. The polynomials can be absorbed in the classical part of the prepotential. We conclude that, as promised in section 2.1, the perturbative prepotential is the limit of the full prepotential under $\mu \to 0$.

Proposition 3.19. *[15] In the limit $\mu \to 0$, the full A_N prepotential can be written as*

$$\mathcal{F} = \mathcal{F}_{class} + \mathcal{F}_{pert} + \sum_{k=1}^{\infty} c_k(a)\mu^k \tag{3.4.22}$$

where $\mathcal{F}_{class}$ is independent of μ and polynomial in the a_i, and $\mathcal{F}_{pert}$ equals the perturbative prepotential (3.4.3), also considered in (2.2.3).

3.5 Prym varieties and the Adler-van Moerbeke problem

We recall the Adler-van Moerbeke question posed in section 3.1.2: the periodic Toda chain for Lie algebra $\mathfrak{g}$ has a Lax pair with spectral parameter for any irreducible representation ρ of $\mathfrak{g}$. The flow of the system linearizes on the Jacobian of the spectral curve (3.1.20) for the Lax operator $A(z)$. Since the Toda system itself is representation independent, the question is whether the Liouville torus can be embedded in a natural way in the Jacobians of each of the spectral curves. This question was answered positively for simply laced Lie algebras in sections 3.1.3 and 3.2. There it was shown that there exists a set of $2N$ cycles and N holomorphic forms whose periods over these cycles generate a submatrix of the period matrix. Therefore they make up an abelian subvariety of dimension $N = \text{rank}(\mathfrak{g})$.

To give more background on this Adler-van Moerbeke problem, we discuss in this section its solution purely in terms of representation theory of finite groups. This work was done independently of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, by mathematicians such as Kanev [34], Merindol [52] and especially Donagi [17]. We start by decomposing the Jacobian of a curve equipped with an action of a finite group on it. Then we show that in the case of the Toda system, where the role of the finite group is played by the Weyl group of $\mathfrak{g}$, for any irreducible representation of $\mathfrak{g}$ there is an abelian variety occurring in the decomposition of the Jacobian variety of the spectral curve. This abelian subvariety is called the distinguished Prym and it solves the Adler-van Moerbeke problem and is therefore the same as the one which defines the prepotential.

3.5.1 Decomposition of Jacobians with finite group action

First we recall some elements of Riemann surface theory. Given a Riemann surface Σ of genus g, a divisor of degree d is a formal linear combination $\sum_i n_i P_i$ of points P_i in Σ and with numbers $n_i \in \mathbf{Z}$ such that $\sum_i n_i = d$. A principal divisor is a divisor of degree zero which consists precisely of the zeroes and poles of a meromorphic function, counting their multiplicity. There is an equivalence relation between divisors, two divisors being equivalent if their formal difference $\sum_i m_i P_i$ (which is a divisor of degree zero) is a principal divisor. This makes the divisor classes of degree zero into an abelian group which by the Jacobi inversion theorem is isomorphic to the Jacobian of Σ. The isomorphism itself is given by the map

$$\sum_i n_i P_i \to \sum_i n_i \left(\int_{P_0}^{P_i} \omega_1, ..., \int_{P_0}^{P_i} \omega_g \right) \tag{3.5.1}$$

where P_0 is a fixed point of Σ. Now given a double cover $f : \tilde{\Sigma} \to \Sigma$ of Riemann surfaces, one can construct a corresponding Norm map N_f on their Jacobians by

$$N_f \left(\sum_i n_i P_i \right) = \sum_i n_i f(P_i) \tag{3.5.2}$$

The Norm map is surjective and the connected component of its kernel containing the identity element is an abelian subvariety of $Jac(\tilde{\Sigma})$.

Definition 3.20. *A classical Prym variety is the connected component containing the identity of the kernel of the Norm map $N_f : Jac(\tilde{\Sigma}) \to Jac(\Sigma)$ belonging to a double cover $f : \tilde{\Sigma} \to \Sigma$ of Riemann surfaces. In terms of the genera $\tilde{g}, g$ of the Riemann surfaces the Prym variety has genus $\tilde{g} - g$.*

The Jacobian $Jac(\tilde{\Sigma})$ splits into two parts: the preimage of $Jac(\Sigma)$ and the Prym. An alternative way of looking at this splitting is to consider the involution σ on $\tilde{\Sigma}$ that is a result of the double cover. This involution acts on the space of holomorphic differentials $\Omega(\tilde{\Sigma})$ by a reducible representation which splits into irreducible ones

$$\Omega(\tilde{\Sigma}) = M_1 \otimes I \oplus M_2 \otimes \epsilon \tag{3.5.3}$$

Here I denotes the 1-dimensional trivial representation, ϵ denotes the also 1-dimensional sign representation and the M_i are the multiplicity spaces, counting how many times each of the two irreducible representations occur in $\Omega(\tilde{\Sigma})$. This is the so-called isotypical decomposition of $\Omega(\tilde{\Sigma})$ into a direct sum of two spaces of dimensions M_1 and M_2. The space $\Omega(\tilde{\Sigma})$ can of course be decomposed further into a direct sum of 1-dimensional subspaces but this decomposition is not canonical. The previously found splitting of the Jacobian variety of $\tilde{\Sigma}$ corresponds precisely with (3.5.3), leading to an alternative definition of the Prym variety as the part of $Jac(\tilde{\Sigma})$ which corresponds with the sign representation. This interpretation leads to the definition of generalized Prym varieties.

Definition 3.21. *Given a (Galois) cover $\tilde{\Sigma} \to \tilde{\Sigma}/G$ for a finite group G acting on $\tilde{\Sigma}$, the space of holomorphic differentials has a decomposition*

$$\Omega(\tilde{\Sigma}) = \oplus_i M_i \otimes V_i \tag{3.5.4}$$

into isotypic pieces coming from the irreducible representations of the finite group G. Since the space of holomorphic differentials can be identified with the tangent space of the Jacobian, the Jacobian also has an isotypic decomposition[6]

$$Jac(\tilde{\Sigma}) \sim \oplus_i Prym_i \otimes V_i \tag{3.5.5}$$

6 This is also called Poincare's irreducibility theorem with G-action [57].

where the $Prym_i \otimes V_i$ are subvarieties of the Jacobian. The $Prym_i$ are called generalized Prym varieties.

Note that the generalized Prym variety for the trivial representation corresponds with $Jac(\tilde{\Sigma}/G)$. The decomposition (3.5.5) is not quite an isomorphism but rather an isogeny. This means that there is a surjective map from $Jac(\tilde{\Sigma})$ to $\oplus_i Prym_i \otimes V_i$ with finite kernel. Isogenies give rise to an equivalence relation between abelian varieties and (3.5.5) states that the two abelian varieties on the left and right hand side are within the same equivalence class. Due to the equivalence relation there has to be an isogeny going back, i.e. a map $\mu : \oplus_i Prym_i \otimes V_i \to Jac(\tilde{\Sigma})$ which is surjective and has finite kernel. This map is given by the sum map

$$\mu(v_1, ..., v_N) = v_1 + ... + v_N \tag{3.5.6}$$

where $+$ denotes the group operation on the abelian variety $Jac(\tilde{\Sigma})$.

3.5.2 Spectral and parabolic covers

Given an arbitrary representation ρ of $\mathfrak{g}$, we can consider the spectral curve Σ_ρ defined in (3.1.20). Its Jacobian will decompose for several reasons: first of all, the representation is in general reducible and this causes the curve to be reducible as well. Secondly, the weights of the irreducible subrepresentations ρ_i come in Weyl orbits and each of these orbits gives a connected component Σ_λ of the spectral curve, labeled by the highest weight λ in the orbit. Finally, the Weyl group acts on Σ_λ and as explained in the previous section, this causes the Jacobian to split. One of the problems in finding the isotypic components of this decomposition is that there are infinitely many Σ_λ to consider. However, it is shown in [17] that they fall into finitely many birational equivalence classes. We will now describe how this comes about.

First we find another way of looking at the Σ_λ. Given the Lie algebra $\mathfrak{g}$ and one of it weights, we construct a cover $\mathfrak{g}_\lambda$ of $\mathfrak{g}$ as follows. Recall that Chevalley's theorem states that there is an isomorphism between the G-invariant polynomials on the Lie algebra $\mathbf{C}[\mathfrak{g}]^G$ and the Weyl invariant polynomials on the Cartan subalgebra $\mathbf{C}[\mathfrak{h}]^W$. This implies that there is a unique G-invariant polynomial

$$P_\lambda : \mathfrak{g} \to \mathbf{C}[x] \tag{3.5.7}$$

whose restriction to the Cartan subalgebra is the Weyl invariant polynomial

$$\prod_{\mu \in W\lambda} (x - \mu) \tag{3.5.8}$$

This allows us to construct a cover $\mathfrak{g}_\lambda$ of the Lie algebra $\mathfrak{g}$ for each weight λ by

$$\mathfrak{g}_\lambda = \{(g, x) \in \mathfrak{g} \times \mathbf{C} | P_\lambda(g)(x) = 0\} \tag{3.5.9}$$

Since there is a map A from $\mathbf{P}^1$ to $\mathfrak{g}$ taking z to its image $A(z)$ defined in (3.1.22) we can pull back the cover $\mathfrak{g}_\lambda$ to a cover Σ_λ of $\mathbf{P}^1$.

Definition 3.22. *Given a simple Lie algebra $\mathfrak{g}$, one of its weights λ and the map $A : \mathbf{P}^1 \to \mathfrak{g}$ we define the spectral cover Σ_λ as*

$$\Sigma_\lambda = \left\{(x, z) \in \mathbf{C}^2 | P_\lambda(A(z))(x) = 0\right\} \tag{3.5.10}$$

In other words, in case $A(z)$ is regular semisimple one can take one of its conjugates $v(z)$ in the Cartan subalgebra and consider Σ_λ to be the curve (3.1.24). In contrast to the infinite number of spectral covers, we will now discuss a single cover called the cameral cover together with rational surjective maps to each of the spectral covers. Consider a simple Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$. The Weyl group W of $\mathfrak{g}$ acts on $\mathfrak{h}$ and we can construct the cover $\pi_1 : \mathfrak{h} \to \mathfrak{h}/W$. One can take the semisimple part of an arbitrary element $g \in \mathfrak{g}$ into $\mathfrak{h}$ by conjugation, and we subsequently take its image under π_1 to define the map $\pi_2 : \mathfrak{g} \to \mathfrak{h}/W$.

Definition 3.23. *The cover $\hat{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ is defined by*

$$\hat{\mathfrak{g}} = \{(g,h) \in \mathfrak{g} \times \mathfrak{h} | \pi_2(g) = \pi_1(h)\} \tag{3.5.11}$$

So a generic fibre over a point $g \in \mathfrak{g}$ consists of a number of $h \in \mathfrak{h}$, one for each Weyl chamber, which is conjugate to g.

This cover has an action of the Weyl group on it. We can again use the map $A : \mathbf{P}^1 \to \mathfrak{g}$ to construct the pull-back $\hat{\Sigma}$ of $\hat{\mathfrak{g}}$, which is a cover of $\mathbf{P}^1$ called the cameral cover.

Definition 3.24. *The cameral cover $\hat{\Sigma} \to \mathbf{P}^1$ is defined by*

$$\hat{\Sigma} = \left\{((g,h),z) \in \hat{\mathfrak{g}} \times \mathbf{P}^1 | A(z) = g\right\} \tag{3.5.12}$$

A generic fibre of this cover again consists of an element in each of the Weyl chambers, hence the name cameral cover.

Since the Weyl group acts on the cameral cover, its Jacobian also has an isotypic decomposition. In an attempt to find the decomposition for the infinite number of spectral covers from the cameral cover, we will construct a natural rational map from $\hat{\Sigma}$ to Σ_λ for each λ. If this map were to be birational, then the Jacobians of the spectral covers would all have the same isotypic decomposition. We will find that this is not the case, and then we go on to construct a finite number of subcovers Σ_P of the cameral cover together with birational maps from a spectral cover Σ_λ to a corresponding Σ_P. Finally, we obtain the decomposition of the Jacobians of the Σ_P from the decomposition for the cameral cover.

Consider for each weight λ of $\mathfrak{g}$ the map $j_\lambda : \hat{\mathfrak{g}} \to \mathfrak{g} \times \mathbf{C}$ given by $j_\lambda(g,h) = (g, \lambda(h))$. We find that $P_\lambda \circ j_\lambda = 0$ because

$$P_\lambda(j_\lambda(g,h)) = P_\lambda(g,\lambda(h)) = P_\lambda(g)(\lambda(h)) = \prod_{\mu \in W\lambda} (x - \mu(t))|_{x=\lambda(t)} = 0 \tag{3.5.13}$$

where we have conjugated g to obtain a Cartan subalgebra element. Since $P_\lambda \circ j_\lambda = 0$ this gives a surjective map $j_\lambda : \hat{\mathfrak{g}} \to \mathfrak{g}_\lambda$ which can be pulled back using A to a covering map $J_\lambda : \hat{\Sigma} \to \Sigma_\lambda$. Therefore the cover $\hat{\Sigma}$ contains all the Σ_λ. This rational surjective map is however not birational because $\hat{\Sigma}$ is too big. We will see that certain intermediate covers $\hat{\Sigma} \to \Sigma_P \to \mathbf{P}^1$ are small enough to give birational maps. The Σ_P will be defined for each parabolic subgroup P of the Weyl group W, thus giving a finite number of birational equivalence classes of spectral curves.

The Weyl group is generated by reflections in a set of simple roots Δ. A parabolic subgroup W_P of the Weyl group is a subgroup generated by reflections in a subset Δ' of Δ. Taking an arbitrary weight λ of $\mathfrak{g}$, its stabilizer group is a parabolic subgroup W_P. Defining the subcover $\mathfrak{g}_P = \hat{\mathfrak{g}}/W_P$ of $\mathfrak{g}$, it is easy to see that the map $j_\lambda : \hat{\mathfrak{g}} \to \mathfrak{g}_\lambda$ factors through $\mathfrak{g}_P$ since λ is invariant under W_P. We thus arrive at the restriction map $j_\lambda : \mathfrak{g}_P \to \mathfrak{g}_\lambda$. Pulling back with A we obtain a surjective rational map $J_\lambda : \Sigma_P \to \Sigma_\lambda$ between covers of $\mathbf{P}^1$. The original map J_λ could not be birational because the generic fibre of $\hat{\Sigma}$ over a point $z \in \mathbf{P}^1$ contains $|W|$ points and that of Σ_λ contains less. But the restricted map can be (and in fact is) birational since the number of points in a generic fibre is equal.

3.5.3 Accidental singularities

Now that we have found a birational map $J_\lambda : \Sigma_P \to \Sigma_\lambda$, we wonder when this map is an isomorphism. That this can happen was shown for $\mathfrak{g} = A_N$ in the fundamental representation in section 3.1.2, where it was shown that Σ_λ is smooth and simply parametrizes the eigenvalues of $A(z)$. In case Σ_λ is singular, J_λ cannot be an isomorphism because Σ_P is smooth. As we already noted in section 3.1.2, Σ_λ will become singular whenever λ and $w\lambda$ accidently take the same values for some regular semisimple element $A(z)$. These singularities are called accidental and their occurrence depends more on the weight λ than on the particular map $A(z)$. Following this reasoning, one obtains the following lemma.

Lemma 3.25 ([17]). *For $J_\lambda : \Sigma_P \to \Sigma_\lambda$ to be an isomorphism, a necessary condition is that λ is the multiple of a fundamental weight.*

So in fact J_λ is usually not an isomorphism because Σ_λ is singular in regular semisimple points. So our assumption made below (3.1.24) on the smoothness of Σ_λ is not very reasonable. Here we see how to amend it: we should consider its natural desingularization Σ_P instead.

3.5.4 The distinguished Prym

We have argued why it's better to study the smooth parabolic covers Σ_P instead of the possibly singular spectral covers Σ_λ. However, the parabolic covers no longer have a natural action of the Weyl group W acting on them. So we cannot think of the splitting of its Jacobian as the result of the Weyl group acting on the cover. However, by letting the parabolic Weyl subgroup W_P act on $\hat{\Sigma}$ and dividing it out to obtain Σ_P, we obtain the map $\pi_P : \hat{\Sigma} \to \Sigma_P$. We can use this map to pull back $Jac(\Sigma_P)$ to $Jac(\hat{\Sigma})$ and intersect it with the isotypic components of $Jac(\hat{\Sigma})$. To see what happens, we can look at this intersection on the level of the tangent spaces, i.e. the spaces of holomorphic differentials $\Omega(\hat{\Sigma})$ and $\Omega(\Sigma_P)$. The holomorphic differentials on Σ_P pull back to W_P invariant holomorphic differentials on $\hat{\Sigma}$, thus showing that the intersection of $Jac(\Sigma_P)$ with $Prym_i \otimes V_i$ is $Prym_i \otimes (V_i)^{W_P}$ where $(V_i)^{W_P}$ denotes the W_P invariant subspace of V_i. This way one obtains a decomposition

$$Jac(\Sigma_P) \sim \oplus_i Prym_i \otimes (V_i)^{W_P} \tag{3.5.14}$$

We now come back to the Adler-van Moerbeke question whether an abelian subvariety of fixed genus sits inside the Jacobian of all spectral curves. If such an abelian subvariety exists we will call it universal. We now see that this question is reduced to a question in finite goup theory, namely if there exists an irreducible representation ρ_i of W such that the dimension of $(V_i)^{W_P}$ is nonzero. This question was answered in [40],[41] and is formulated as follows.

Theorem 3.26. *The generalized Prym variety corresponding with the reflection representation of the Weyl group has genus equal to rank*$(\mathfrak{g})$ *and occurs with nonzero multiplicity inside the Jacobians of all spectral curves, therefore it is universal. Moreover, this is the only universal Prym variety if and only if* $\mathfrak{g}$ *is a classical Lie algebra.*

This abstract formulation of the answer to the Adler-van Moerbeke problem is consistent with the more concrete result found earlier in section 3.1.5. Instead of letting the Weyl group act on the space of holomorphic differentials, we there had the dual picture where it acts on the space of cycles. The special subset of A cycles constructed there precisely transforms under the reflection representation of the Weyl group, thus showing that it indeed occurs with nonzero multiplicity.

3.5.5 The distinguished Prym as a Jacobian

The whole problem of the Jacobian of the spectral curve being too big to be the Liouville torus can be avoided if the distinguished Prym variety (i.e. the Liouville torus) is itself the Jacobian of some smaller curve. Indeed, the Prym is principally polarized so it very well could be a Jacobian, but on the other hand it is known that not all Pryms are Jacobians [55]. For classical Lie algebras, the Pryms are indeed Jacobians. For type A_N this is trivial because the spectral curve for the fundamental representation has genus N, and the spectral curves (3.1.28) for the other classical groups in the fundamental representation have an involution $x \to -x$ besides the hyperelliptic involution $y \to -y$. Dividing out certain combinations of these involutions gives the genus N Prym variety as the Jacobian of a curve [17]. For other Lie algebras, this rather ad hoc method involves trying to recognize involutions of spectral curves for the simplest representation and figuring out whether or not the distinguished Prym occurs as the Jacobian of one of the quotients. That this does not always work is shown in the example of Lie algebra G_2, also discussed in [17].

Let us recall the genus 11 spectral curve for G_2 in the smallest representation

$$3\left(z-\frac{\mu}{z}\right)^2 - x^8 + 2ux^6 - \left[u^2 + z + \frac{\mu}{z}\right]x^4 + \left[v + 2u\left(z + \frac{\mu}{z}\right)\right]x^2 = 0 \tag{3.5.15}$$

We introduce the variables

$$\begin{aligned} w &= z + \frac{\mu}{z} \\ r &= x^2 \\ s &= x\left(z - \frac{\mu}{z}\right) \\ t &= x\left(w - r^2 + \frac{v}{3}\right) \end{aligned} \tag{3.5.16}$$

which occur when one starts to divide out the symmetries of the curve. The curve written in terms of the variables r, t has genus two, and so it was suggested in [17] that its Jacobian is the distinguished Prym. The curve is

$$P(r,t) = 3t^2 - 4r^5 + 4r^4u - \frac{4}{3}r^3u^2 - 12r\mu + r^2v = 0 \tag{3.5.17}$$

and its holomorphic differentials are given by

$$
\begin{aligned}
\omega_1 &= \frac{dt}{P_r} \\
\omega_2 &= \frac{r dt}{P_r}
\end{aligned}
\tag{3.5.18}
$$

and it is readily seen that these do not correspond to the derivatives of the Seiberg-Witten differential (3.1.37). Therefore the Jacobian of this genus two curve is not the Liouville torus of the Toda system for G_2. The question whether or not the distinghuised Prym is a Jacobian for all simple Lie algebras remains unanswered.

Appendix A

The Seiberg-Witten curves for $\mathbf{E}_6$ and $\mathbf{F}_4$

Appendix A

This appendix contains data on the family of curves for Lie algebras E_6 and F_4. The E_6 curve reads

$$P_{E_6} = \frac{1}{2}x^3(z + \frac{\mu}{z} + u_6)^2 - q_1(x)(z + \frac{\mu}{z} + u_6) + q_2(x) = 0$$

where the polynomials q_1 and q_2 are given by

$$\begin{aligned}
q_1 &= 270x^{15} + 342u_1x^{13} + 162u_1^2x^{11} - 252u_2x^{10} + (26u_1^3 + 18u_3)x^9 \\
&\quad -162u_1u_2x^8 + (6u_1u_3 - 27u_4)x^7 - (30u_1^2u_2 - 36u_5)x^6 + \\
&\quad (27u_2^2 - 9u_1u_4)x^5 - (3u_2u_3 - 6u_1u_5)x^4 - 3u_1u_2^2x^3 - 3u_2u_5x - u_2^3, \\
q_2 &= \frac{1}{2x^3}(q_1^2 - p_1^2p_2), \\
p_1 &= 78x^{10} + 60u_1x^8 + 14u_1^2x^6 - 33u_2x^5 + \\
&\quad 2u_3x^4 - 5u_1u_2x^3 - u_4x^2 - u_5x - u_2^2, \\
p_2 &= 12x^{10} + 12u_1x^8 + 4u_1^2x^6 - 12u_2x^5 + \\
&\quad u_3x^4 - 4u_1u_2x^3 - 2u_4x^2 + 4u_5x + u_2^2.
\end{aligned}$$

The curve for F_4 on the other hand reads

$$P_{F_4} = -8\left(z + \frac{\mu^2}{z}\right)^3 + s_1(x)\left(z + \frac{\mu^2}{z}\right)^2 + s_2(x)\left(z + \frac{\mu^2}{z}\right) + s_3(x) = 0$$

where the $s_i(x)$ are given by

$$\begin{aligned}
s_1(x) &= -636x^9 - 300u_1x^7 - 48{u_1}^2x^5 - 5u_3x^3 + 2u_4x, \\
s_2(x) &= -168x^{18} - 348u_1x^{16} - 276{u_1}^2x^{14} + (-116{u_1}^3 + 14u_3)x^{12} \\
&\quad +(-92u_4 - 20{u_1}^4 - 8u_1u_3)x^{10} + (-42u_1u_4 - 6{u_1}^2u_3)x^8 \\
&\quad +(-4u_6 - \frac{10}{3}{u_1}^2u_4 - \frac{2}{3}{u_3}^2)x^6 + (\frac{1}{3}u_3u_4 - \frac{2}{3}u_6u_1)x^4, \\
s_3(x) &= x^{27} + 6u_1x^{25} + 15{u_1}^2x^{23} + (20{u_1}^3 + u_3)x^{21} + \\
&\quad (5u_4 + 4u_1u_3 + 15{u_1}^4)x^{19} + (6{u_1}^2u_3 + 12u_1u_4 + 6{u_1}^5)x^{17} + \\
&\quad (\frac{1}{3}{u_3}^2 + 5u_6 + 4{u_1}^3u_3 + \frac{26}{3}{u_1}^2u_4 + {u_1}^6)x^{15} \\
&\quad +(\frac{4}{3}{u_1}^3u_4 + \frac{19}{3}u_6u_1 + {u_1}^4u_3 + \frac{4}{3}u_3u_4 + \frac{2}{3}{u_3}^2u_1)x^{13} \\
&\quad +(\frac{1}{3}{u_1}^2{u_3}^2 - \frac{1}{3}{u_1}^4u_4 - \frac{15}{4}{u_4}^2 + 3u_6{u_1}^2)x^{11} \\
&\quad +(\frac{1}{3}u_6u_3 - \frac{4}{9}{u_1}^2u_3u_4 + \frac{1}{27}{u_3}^3 - \frac{13}{6}{u_4}^2u_1 + \frac{13}{27}u_6{u_1}^3)x^9 \\
&\quad +(-\frac{1}{9}{u_3}^2u_4 - \frac{1}{2}u_6u_4 + \frac{1}{9}u_6u_1u_3 - \frac{7}{36}{u_1}^2{u_4}^2)x^7 + \\
&\quad (\frac{1}{12}{u_4}^2u_3 - \frac{1}{6}u_6u_1u_4)x^5 + (-\frac{1}{54}{u_4}^3 - \frac{1}{108}{u_6}^2)x^3.
\end{aligned}$$

Samenvatting

Er bestaat tot op heden geen algemene theorie voor het vinden van exacte oplossingen van stelsels van partiële differentiaalvergelijkingen. Soms kan existentie en uniciteit van oplossingen worden aangetoond via de stelling van Cauchy-Kovalevskaya, of kunnen oplossingen expliciet worden geconstrueerd omdat ze tot een bepaalde klasse van vergelijkingen horen, bijvoorbeeld de integreerbare vergelijkingen. In dit proefschrift bekijken we het Witten-Dijkgraaf-Verlinde-Verlinde ofwel WDVV stelsel van partiële differentiaalvergelijkingen en met name een generalisatie daarvan. Dit is een overbepaald stelsel van derde orde niet-lineaire vergelijkingen. In het algemeen mag men niet verwachten dat een overbepaald stelsel nog oplossingen toelaat, maar voor de (gegeneraliseerde) WDVV vergelijkingen is dit toch het geval, wat de vergelijkingen reeds tot bijzondere maakt. De oplossingen die we zullen bekijken en die het hoofdonderwerp van dit proefschrift vormen hebben hun natuurlijke context binnen de zogenaamde Seiberg-Witten theorie, een fysisch model voor het beschrijven van quarks en aanverwante elementaire deeltjes. Het feit dat we expliciete oplossingen kunnen construeren voor een ingewikkeld stelsel als WDVV maakt deze vergelijkingen tot heel bijzondere.

De opbouw van dit proefschrift is als volgt. In hoofdstuk een worden de WDVV vergelijkingen en hun generalisatie geïntroduceerd. Er wordt uitgelegd hoe de vergelijkingen gezien kunnen worden als de conditie om de derde orde afgeleiden van een functie op te vatten als de structuurconstanten van een associatieve commutatieve algebra.

Hoofdstuk twee is één van de centrale hoofdstukken en bevat expliciete oplossingen van zowel de gegeneraliseerde als de oorspronkelijke WDVV vergelijkingen. Deze oplossingen hebben hun directe oorsprong in Seiberg-Witten theorie waar ze bekend staan onder de naam perturbatieve prepotentiaal. Onderscheid wordt gemaakt tussen de vier-dimensionale prepotentialen, die aanleiding geven tot oplossingen van de gegeneraliseerde vergelijkingen, en de vijf-dimensionale prepotentialen die zelfs oplossingen geven van de oorspronkelijke WDVV vergelijkingen. Met name de mogelijkheid om oplossingen te construeren voor elke simpele Lie algebra is opmerkelijk en suggereert een dieper verband tussen de WDVV vergelijkingen en Lie algebras, een verband dat nog niet goed is begrepen.

Hoofdstuk drie is het andere centrale hoofdstuk. Hierin worden de vier-dimensionale niet-perturbatieve prepotentialen beschreven, die ook oplossingen zijn van de gegeneraliseerde WDVV vergelijkingen. Hoewel deze oplossingen niet expliciet kunnen worden gegeven in termen van gesloten uitdrukkingen zijn ze toch zeer interessant. Ten eerste worden ze in een bepaalde limiet gereduceerd tot de perturbatieve prepotentialen en zijn dus daarmee direct verbonden. Maar meer nog dan dat zijn de niet-perturbatieve prepotentialen interessant omdat ze onderdeel zijn van een prachtig geometrisch kader dat de oplossing geeft van Seiberg-Witten theorie. Omdat een uitgebreide beschrijving van de natuurkundige Seiberg-Witten theorie teveel tijd zou vergen hebben we ervoor gekozen om deze te reduceren tot zijn puur wiskundige inhoud. Om toch te kunnen waarderen hoe bijzonder de prepotentialen zijn is besloten ze in te bedden in de context van een integreerbaar systeem: de periodieke Toda ketting. Dit systeem is nauw verwant aan Seiberg-Witten theorie en de constructie van de prepotentiaal geeft het antwoord op een zeer belangrijke vraag voor de Toda ketting, namelijk hoe zijn Liouville torus te beschrijven is.

Curriculum Vitae

Luuk Hoevenaars werd geboren op 17 juni 1975 in Nijmegen. In 1993 behaalde hij op de Nijmeegse Scholengemeenschap Groenewoud zijn VWO-diploma met als vakkenpakket Nederlands, Engels, Natuurkunde, Wiskunde B, Scheikunde, Biologie, Latijn, Grieks en Geschiedenis. Hij is in 1993 begonnen met zijn studie in Utrecht en in 1994 behaalde hij zijn propaedeuse in zowel wiskunde als natuurkunde. In 1998 studeerde hij af in de natuurkunde bij prof. dr. B. de Wit met als onderwerp het massaspectrum in Seiberg-Witten theorie.

In september 1998 begon hij als Assistent in Opleiding (AIO) bij de leerstoel Fundamentele Analyse & Algebra. Onder begeleiding van prof. dr. ir. Ruud Martini en dr. Paul Kersten heeft hij onderzoek verricht naar exacte oplossingen van de Witten-Dijkgraaf-Verlinde-Verlinde vergelijkingen, wat heeft geleid tot dit proefschrift. Tijdens zijn promotie heeft hij voordrachten gehouden en is hij aanwezig geweest op verscheidene lentescholen en congressen. Naast zijn wetenschappelijke werk heeft hij onderwijs gegeven aan de universiteit en was hij faculteitsvertegenwoordiger voor het Twents AIO Beraad.

Publicaties:

L. Hoevenaars. Duality transformation for generalized WDVV equations in Seiberg-Witten theory. *hep-th/0202007*.

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